EGR3320 Industrial Robotics

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Inverse Kinematics

- Up until now , we focused on forward kinematics
  - What is it?

Inverse Kinematics

- Up until now , we focused on forward kinematics
  - We were given the joint variables and we were asked to find the location of the end effector
    - We were deriving transformation matrices
- How about inverse kinematics (what is it)?

Inverse Kinematics

- How about inverse kinematics
  - We are given a desired location of the end effector and we are asked to find the joint variables to result in that end effector location
  - Very useful in robotics=> any examples?

Inverse Kinematics

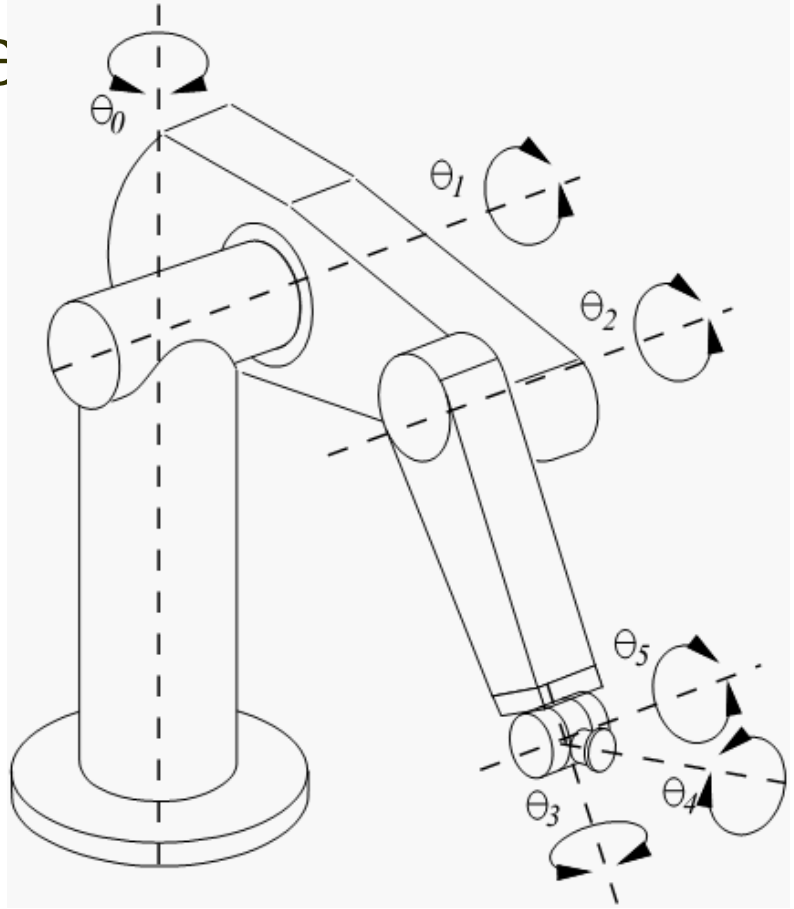
- Inverse kinematics
  - About the problem
  - Method of solution
  - Example

About the problem

- We want to compute the set of ***joint variables***, given the desired position and orientation of the tool relative to the station.
- The problem is ***non-linear***: given ${}^{O}_{n}\boldsymbol{T}$, find the values of $\boldsymbol{\theta_1}, \ldots, \boldsymbol{\theta_n}$.
  These equations can be ***non-linear***, ***transcendental***.

Example: PUMA 560

- Frames and link parameters (direct kine



Example: PUMA 560

- The kinematics equations of the PUMA 560:

$$ {}^0_6T = {}^0_1T\,{}^1_2T\,{}^2_3T\,{}^3_4T\,{}^4_5T\,{}^5_6T $$

$$ = \begin{pmatrix} r_{11} & r_{12} & r_{13} & p_X \\ r_{21} & r_{22} & r_{23} & p_Y \\ r_{31} & r_{32} & r_{33} & p_Z \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

With:

$$ p_X = c_1[a_2c_2 + a_3c_{23} - d_4s_{23}] - d_3s_1, $$

$$ p_Y = s_1[a_2c_2 + a_3c_{23} - d_4s_{23}] - d_3c_1, $$

$$ p_Z = -a_3s_{23} - a_2s_2 - d_4c_{23}, $$

$$ r_{11} = c_1[c_{23}(c_5c_6 - s_4s_5) - s_{23}s_5c_5] + s_1(s_4c_5c_6 + c_4s_6), $$

$$ r_{21} = s_1[c_{23}(c_4c_5c_6 - s_4s_6) - s_{23}s_5c_6] + c_1(s_4c_5c_6 + c_4s_6), $$

$$ r_{31} = -s_{23}(c_4c_5c_6 - s_4s_6) - s_{23}s_5c_6, $$

$$ r_{12} = c_1[c_{23}(-c_4c_5c_6 - s_4c_6) + s_{23}s_5c_6] + s_1(c_4c_6 - s_4c_5s_6), $$

$$ r_{22} = s_1[c_{23}(-c_4c_5s_6 - s_4c_6) + s_{23}s_5s_6] - c_1(c_4c_6 - s_4c_5s_6), $$

$$ r_{32} = -s_{23}(-c_4c_5s_6 - s_4c_6) + c_{23}s_5s_6, $$

$$ r_{13} = -c_1(c_{23}c_4s_5 + s_{23}c_5) - s_1s_4s_5, $$

$$ r_{23} = -s_1(c_{23}c_4s_5 + s_{23}c_5) + c_1s_4s_5, $$

$$ r_{33} = s_{23}c_4s_5 - s_{23}c_5. $$

About the problem

- Consider the PUMA 560:
  - Given the matrice ${}^{0}_{6}T$, solve PUMA 560's kinematics equations for the joint angles θ_1 through θ_6.
  - We get 3 independent equations from the rotation-matrix part of ${}^{0}_{6}T$ and 3 equations from the position-vector part of ${}^{0}_{6}T$.
  - Thus: 6 nonlinear, transcendental equations and 6 unknowns, and this for a "very simple" 6 DOFs manipulator!

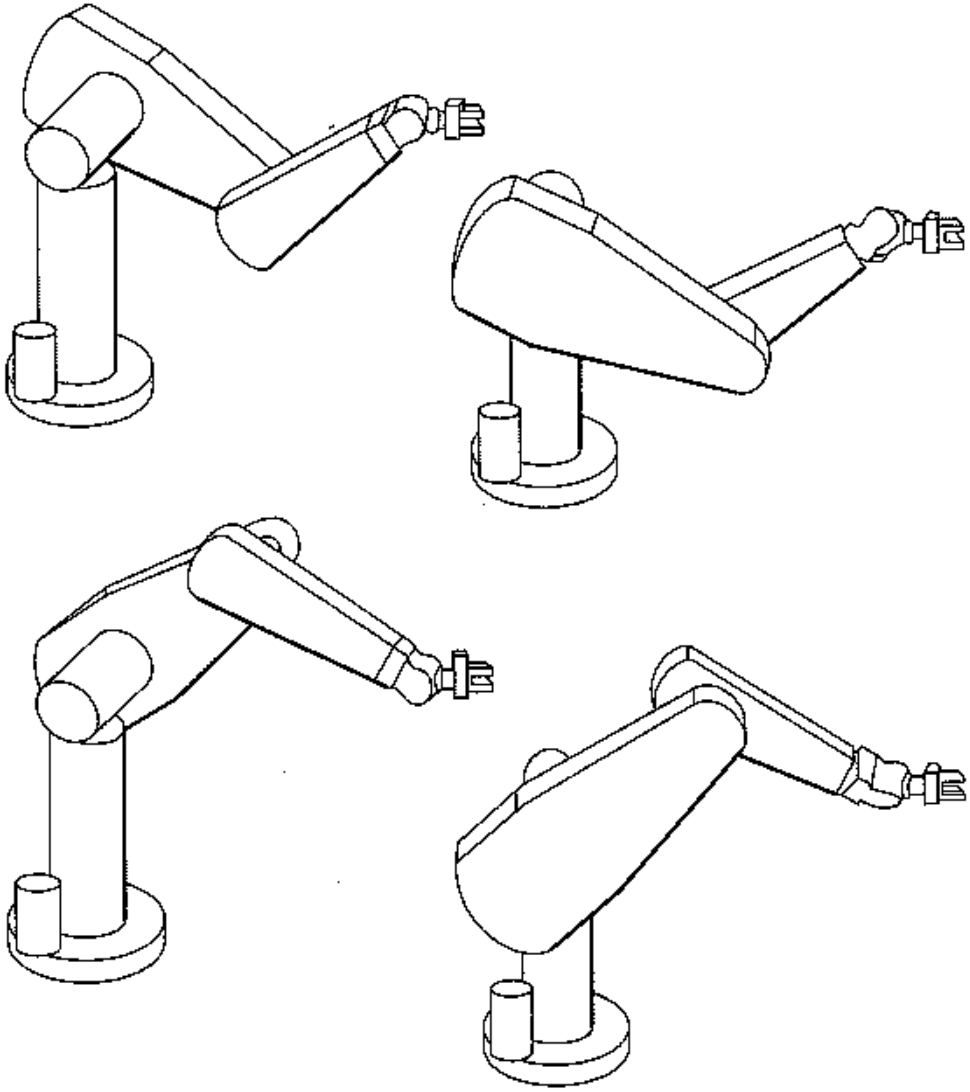
About the problem

- We must ask the following questions:
  - Is there a solution?
  - Are there several solutions?
  - How to solve the problem?

Workspace

- If the ***goal*** (desired position and orientation) is in the ***reachable workspace***, then there's at least 1 solution.
- The ***reachable workspace*** is dependent on the manipulator.
- There might be several solutions.
- We won't cover the following considerations:
  obstacles, limits on joint ranges,...

Multiple solutions



Inverse Kinematics

- Inverse kinematics
  - About the problem
  - Method of solution
  - Example

Method of solution

- There’s no general algorithm
- Consider a manipulator as “***solvable***” if it is possible to calculate ***all*** the solutions.
- Manipulator solution strategies might be split into 2 classes: ***closed-form*** and ***numerical*** solutions (numerical solutions won’t be covered – generally slower because of their iterative nature).

Method of solution

- ***closed-form*** solutions methods are solutions methods based on analytic expressions or on the solution of a polynomial of degree 4 or less.
- There's a general ***numerical solution*** for which ***all "6 DOFs in a single chain" system with revolute and prismatic joints are solvable!***
- Only on special cases can they be solved analytically.

Method of solution

- There are several closed-form solution strategies. We'll discuss the following 2:
  - Algebraic solution by ***reduction to polynomial***
  - ***Pieper's criteria***
- We'll also discuss a heuristically solution strategy:
  - ***Cyclic Coordinate Descent (CCD)***

Algebraic solution by reduction to polynomial

- Substitution:

$$u = \tan\frac{\theta}{2}$$

$$\cos\theta = \frac{1-u^2}{1+u^2}, \sin\theta = \frac{2u}{1+u^2}$$

- Advantage:
  - the substitution yields an expression in terms of variable u_i instead of $\sin\theta_i$ and $\cos\theta_i$.
- Once the solutions for u_i are found, $\theta_i = 2\tan^{-1}(\text{solutions-of-}u_i)$.

Pieper’s criteria

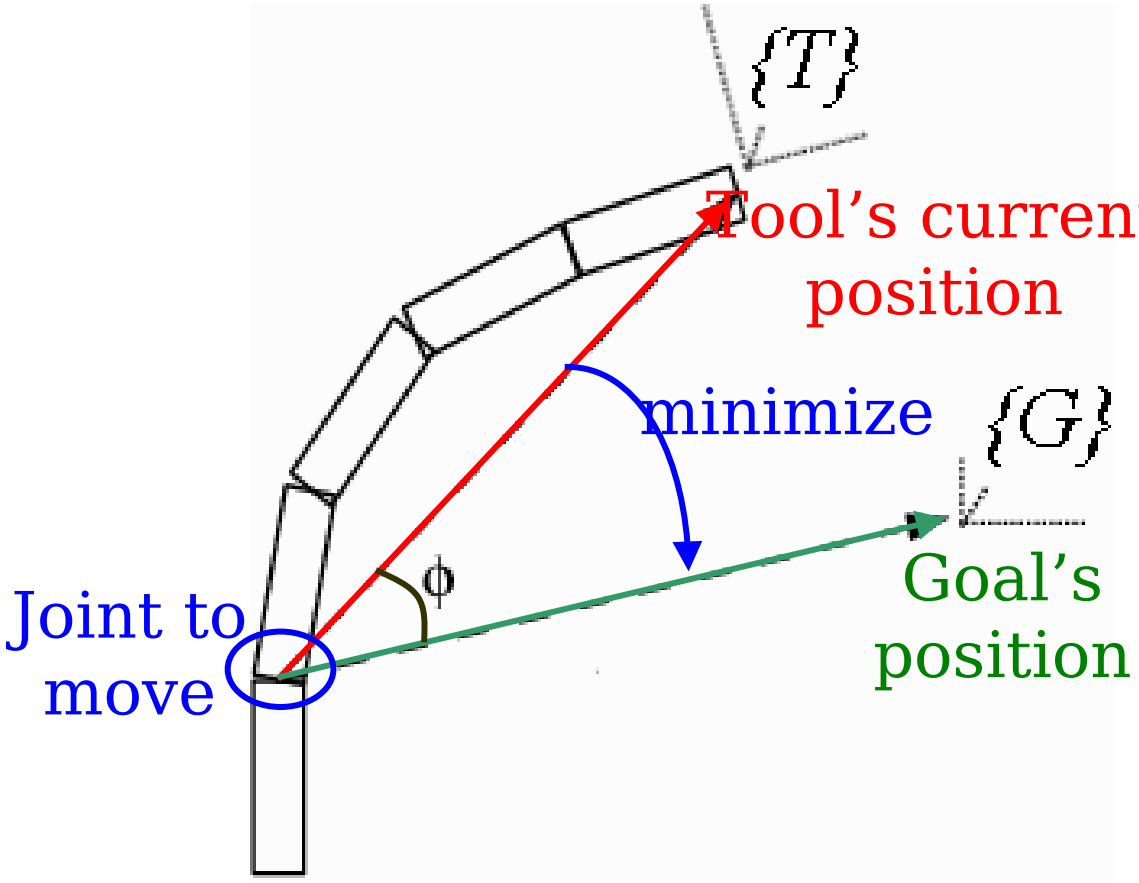
- There’s a closed-form solution for ***6 DOFs manipulators*** (with prismatic and/or revolute joints configurations) in which ***3 consecutive axes intersect in 1 point***.
- Almost every 6 DOFs manipulator built today respect Pieper’s criteria.
- For ex: PUMA 560’s joint axes 4, 5 and 6.

Cyclic Coordinate Descent

- Algorithm:
  - *adjusting one DOF at a time (**iterative**) to minimize tool's distance to the goal*
  - *starts at the last link and works backwards, adjusting each joint along the way*
  - *repeat the whole set until "satisfied" or maximum nr. of sets reached*
- *Each step results in one equation with one unknown for each degree.*

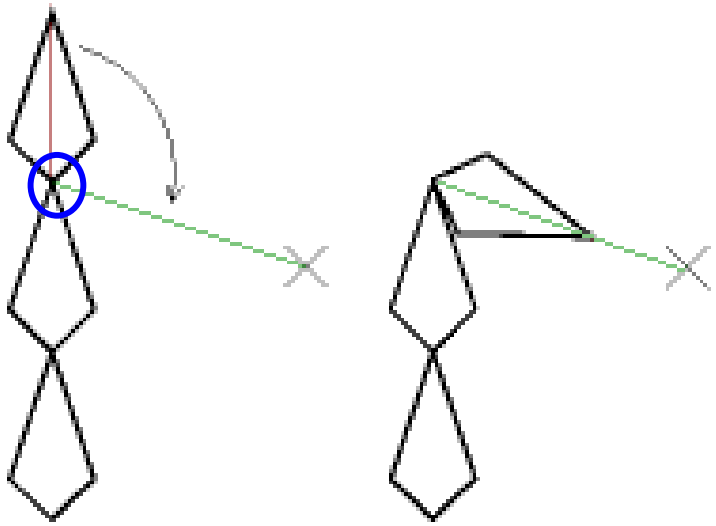
Cyclic Coordinate Descent

- *Adjusting one link at the time*



Cyclic Coordinate Descent

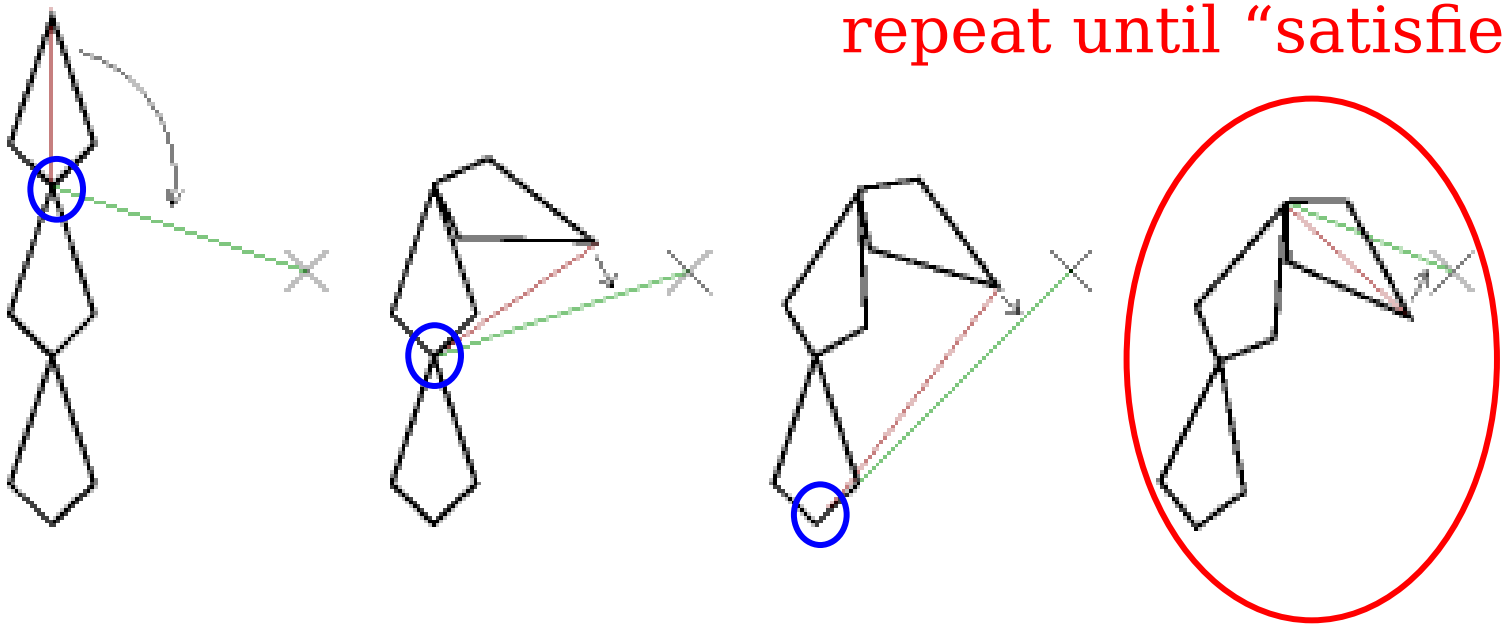
- *Adjusting one link at the time*



Cyclic Coordinate Descent

- *starts at the last link, adjusting each joint along the way*

repeat until “satisfied”



Cyclic Coordinate Descent

Advantages:

- Allow constraints to be placed (at each step)
- Free of singularities
- Degree independent
- Computationally inexpensive (fast)
- Simple to implement

- Disadvantage:
  - Might not find a solution

Inverse Kinematics

- Example

Example: PUMA 560

- We know the links transformations in θ_i ($1 \le i \le 6$).

$$ {}^{0}_{1}T = \begin{pmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad {}^{3}_{4}T = \begin{pmatrix} c\theta_4 & -s\theta_4 & 0 & a_3 \\ 0 & 0 & 1 & d_4 \\ -s\theta_4 & -c\theta_4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ {}^{1}_{2}T = \begin{pmatrix} c\theta_2 & -s\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s\theta_2 & -c\theta_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad {}^{4}_{5}T = \begin{pmatrix} c\theta_5 & -s\theta_5 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s\theta_5 & c\theta_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ {}^{2}_{3}T = \begin{pmatrix} c\theta_3 & -s\theta_3 & 0 & a_2 \\ s\theta_3 & c\theta_3 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad {}^{5}_{6}T = \begin{pmatrix} c\theta_6 & -s\theta_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s\theta_6 & -c\theta_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

Example: PUMA 560

- We know the links transformations in θ_i ($1 \le i \le 6$).
- Thus, we know the transformation from ${}^{k}_{l}T$ to ${}^{l}_{k}T$ in θ ($1 \le k,l \le 6$ and $k \le i \le l$ w.l.o.g.).

$$ {}^{k}_{l}T = {}^{l-1}_{k}T \cdot {}^{l}_{l-1}T \cdot \ldots \cdot {}^{k}_{k+1}T $$

Example: PUMA 560

- We know the links transformations in $\boldsymbol{\theta_i}$ ($\boldsymbol{1 \le i \le 6}$).
- Thus, we know the transformation from $\boldsymbol{\{k\}}$ to $\boldsymbol{\{l\}}$ in $\boldsymbol{\theta_i}$ ($\boldsymbol{1 \le k,l \le 6}$ and $\boldsymbol{k \le i \le l}$ w.l.o.g.).
- We wish to solve:

$$
{}^{0}_{6}T = \begin{pmatrix} r_{11} & r_{12} & r_{13} & p_X \\ r_{21} & r_{22} & r_{23} & p_Y \\ r_{31} & r_{32} & r_{33} & p_Z \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
= {}^{0}_{1}T(\theta_1)\,{}^{1}_{2}T(\theta_2)\,{}^{2}_{3}T(\theta_3)\,{}^{3}_{4}T(\theta_4)\,{}^{4}_{5}T(\theta_5)\,{}^{5}_{6}T(\theta_6)
$$

Example: PUMA 560

- We'll search for a solvable equation iteratively:
  - multiply each side of the transform equation by an inverse to separate a variable
  - Search for a solvable equations

$$ {}^0_6T = {}^0_1T(\theta_1)\,{}^1_2T(\theta_2)\,{}^2_3T(\theta_3)\,{}^3_4T(\theta_4)\,{}^4_5T(\theta_5)\,{}^5_6T(\theta_6) $$

$$ \Leftrightarrow {}^0_1T(\theta_1)^{-1}\,{}^0_6T = {}^1_2T(\theta_2)\,{}^2_3T(\theta_3)\,{}^3_4T(\theta_4)\,{}^4_5T(\theta_5)\,{}^5_6T(\theta_6) $$

$$ \Leftrightarrow \begin{pmatrix} c_1 & s_1 & 0 & 0 \\ -s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r_{11} & r_{12} & r_{13} & p_X \\ r_{21} & r_{22} & r_{23} & p_Y \\ r_{31} & r_{32} & r_{33} & p_Z \\ 0 & 0 & 0 & 1 \end{pmatrix} = {}^1_6T $$

Example: PUMA 560

- After calculating the right side, it can been seen there's a solvable solution:

$$\begin{pmatrix} c_1 & s_1 & 0 & 0 \\ -s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r_{11} & r_{12} & r_{13} & p_X \\ r_{21} & r_{22} & r_{23} & p_Y \\ r_{31} & r_{32} & r_{33} & p_Z \\ 0 & 0 & 0 & 1 \end{pmatrix} = {}^{1}_{6}T$$

- And we get: $-\sin\theta_1 p_X + \cos\theta_1 p_Y = d_3$.
- $a\cos\theta_i + b\sin\theta_i = c$ return 2 solutions:

$$\theta_i = \arctan 2(b,a) \pm \arctan 2(\sqrt{a^2 + b^2 - c^2}, c)$$

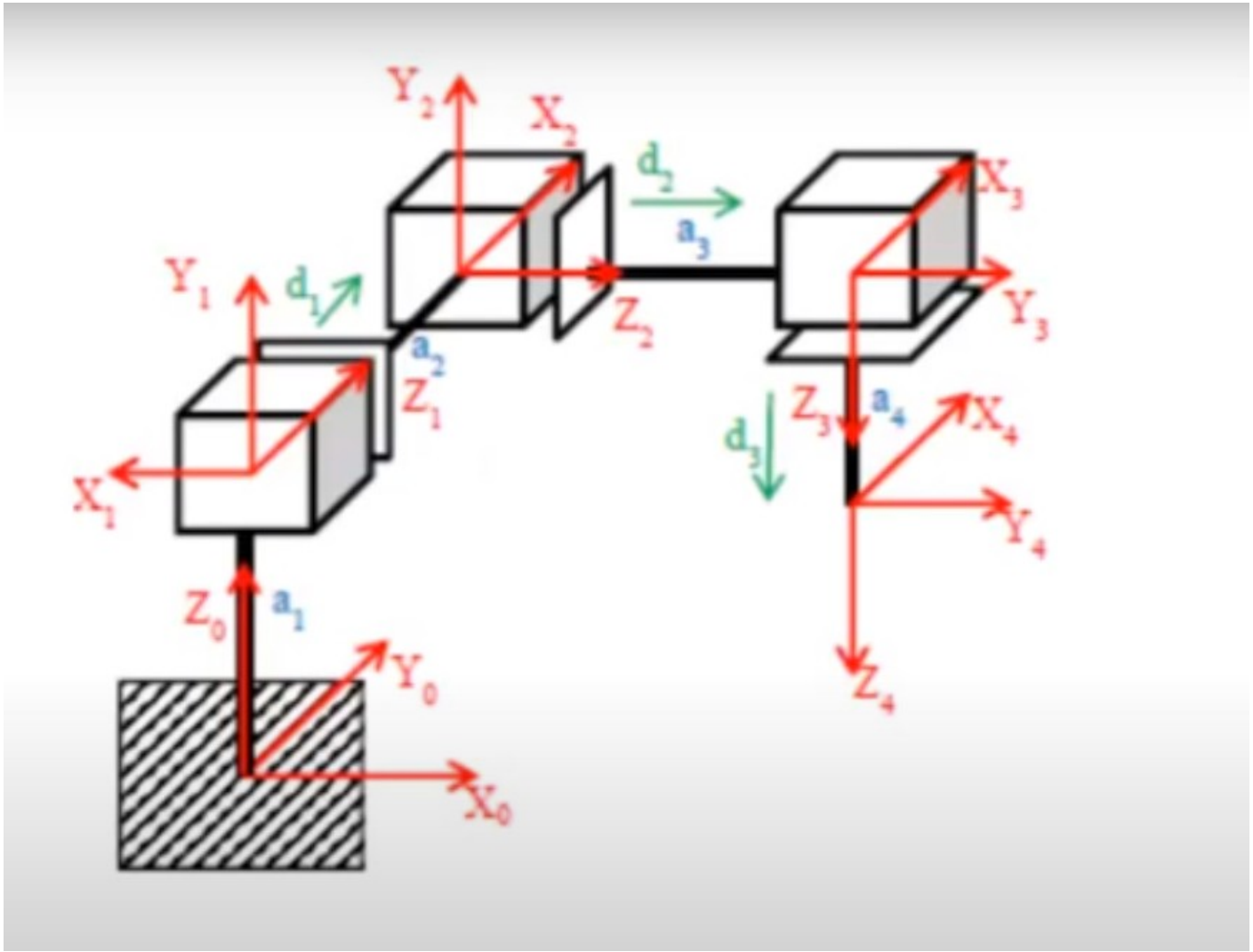
Example: PUMA 560

- 2 solutions were found for $\boldsymbol{\theta_1}$. For each one of them, we'll continue to search for solvable equations...

$$\begin{pmatrix} c_1 & s_1 & 0 & 0 \\ -s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r_{11} & r_{12} & r_{13} & p_X \\ r_{21} & r_{22} & r_{23} & p_Y \\ r_{31} & r_{32} & r_{33} & p_Z \\ 0 & 0 & 0 & 1 \end{pmatrix} = {}^{1}_{6}T$$

Example2: simple 3DOF manipulator

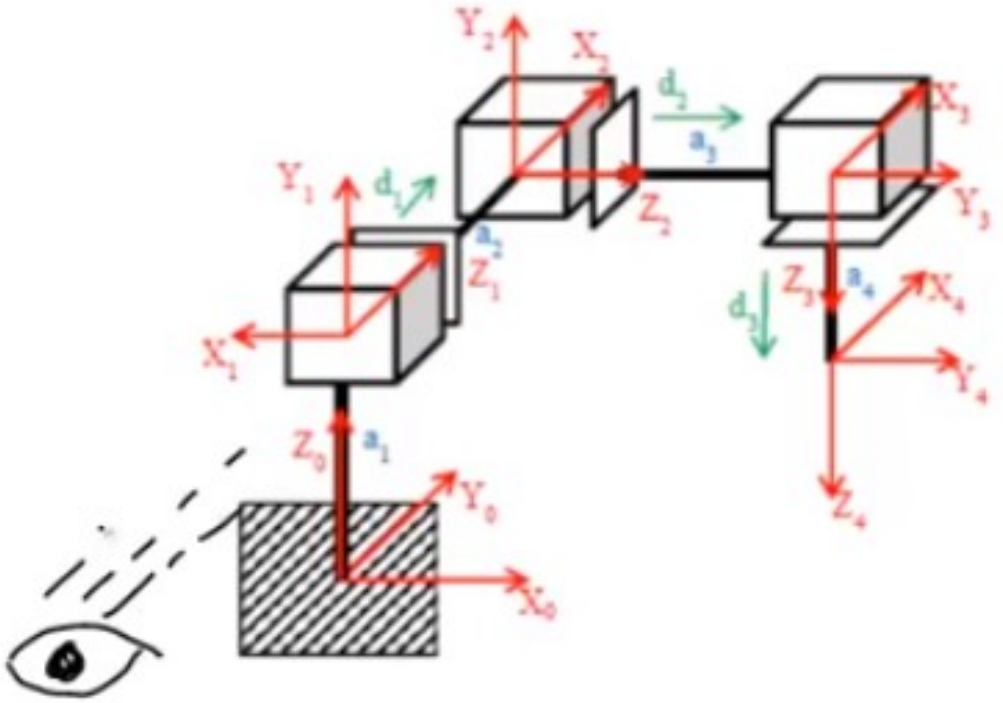
Find the X, Y and Z coordinates of the end effector in terms of the joint .variable, then find the joint variables in terms of these coordinates



Example2: simple 3DOF manipulator

Find the X, Y and Z coordinates of the end effector in terms of the joint variable, then find the joint variables (d1, d2 and d3)in terms of these .coordinates

First, draw the side view



Example2: simple 3DOF manipulator

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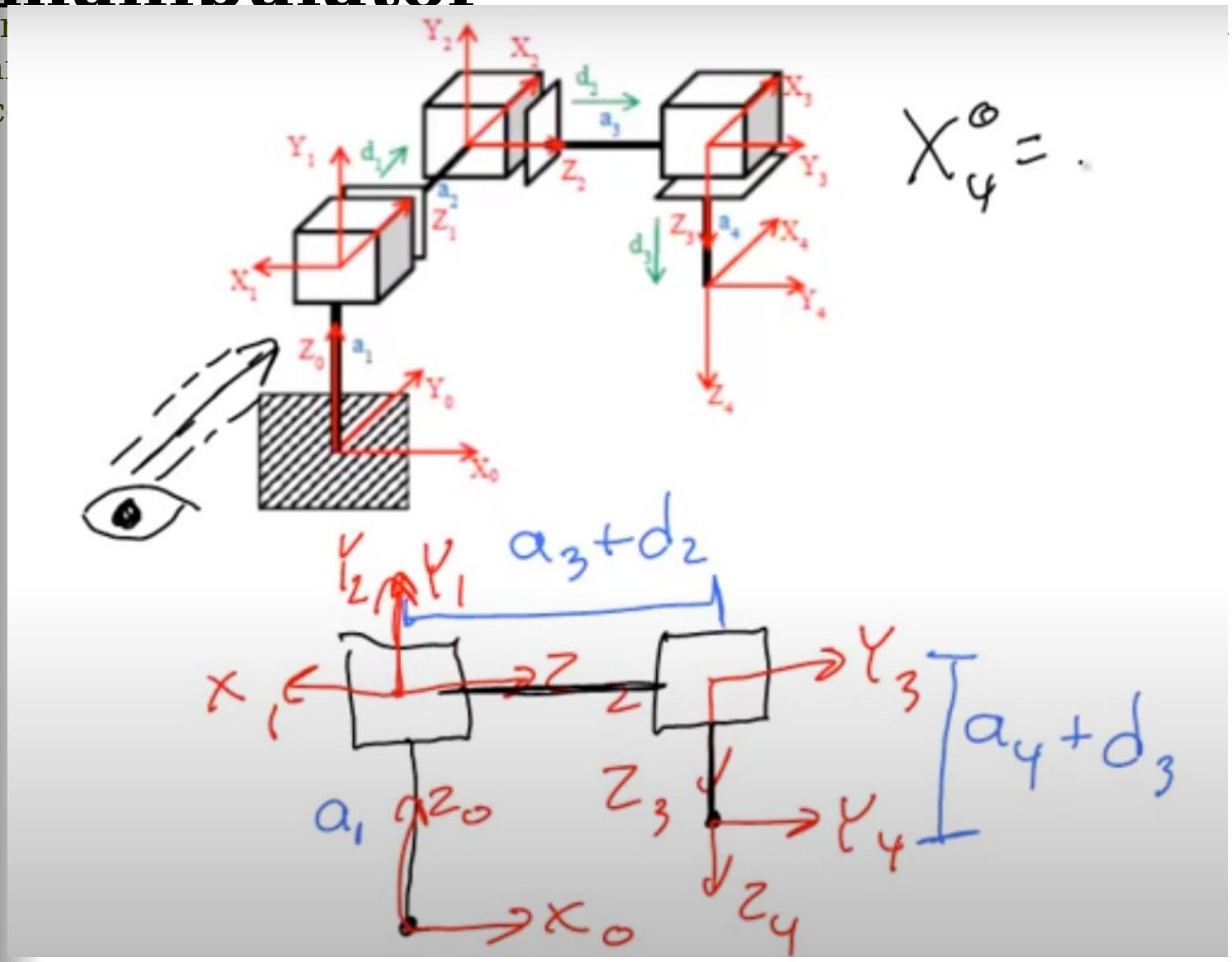

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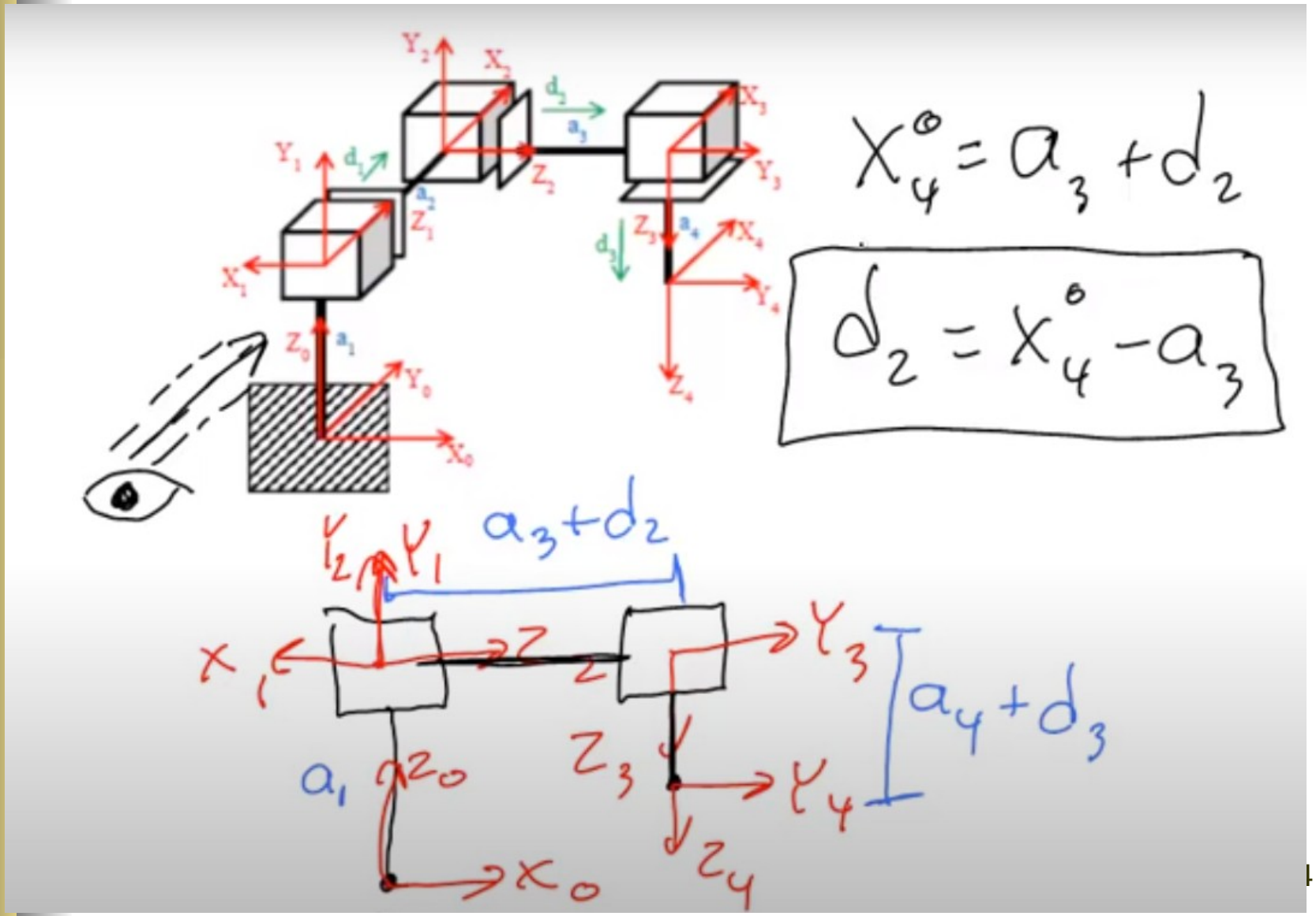
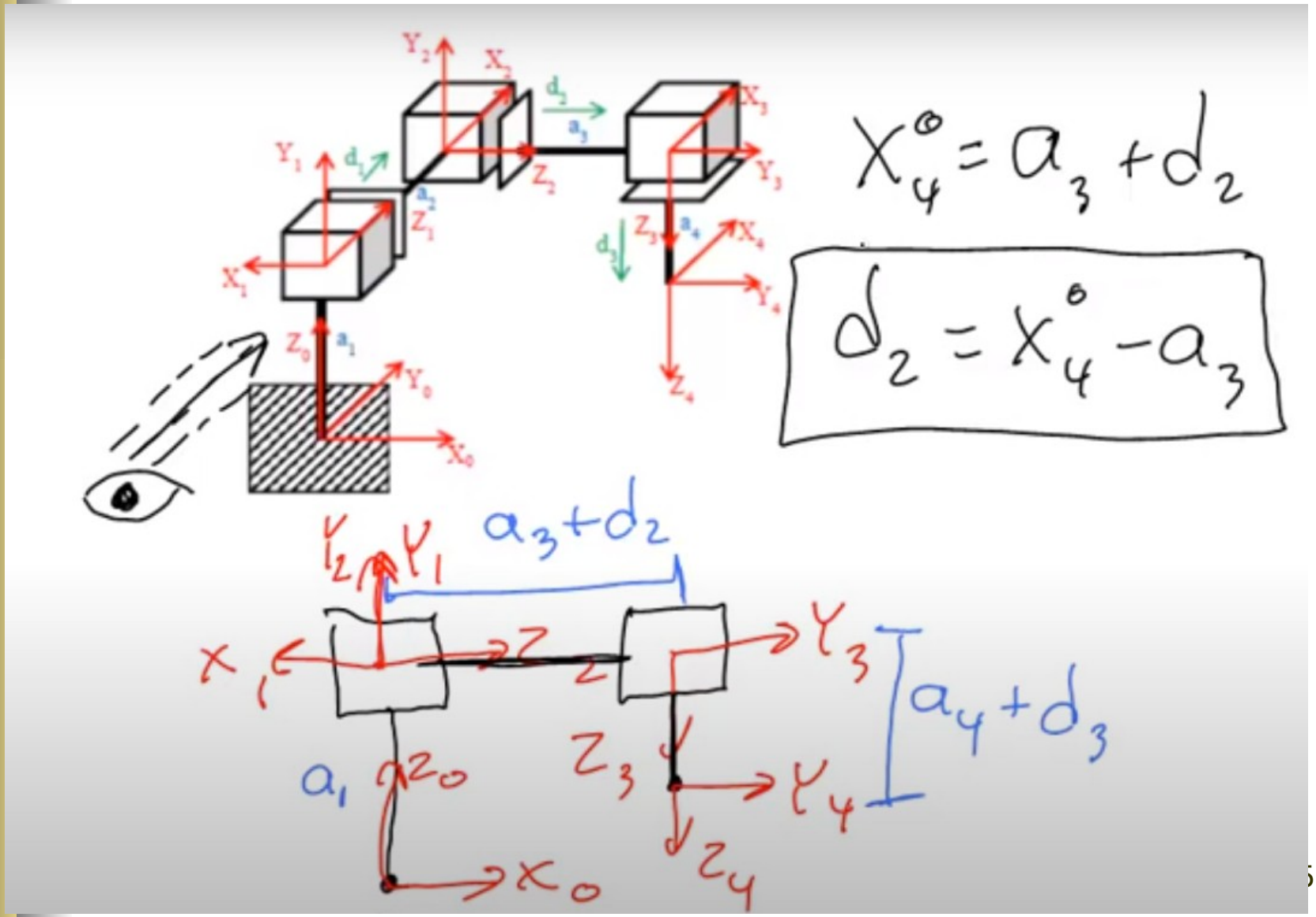


Example2: simple 3DOF manipulator

?How about d3



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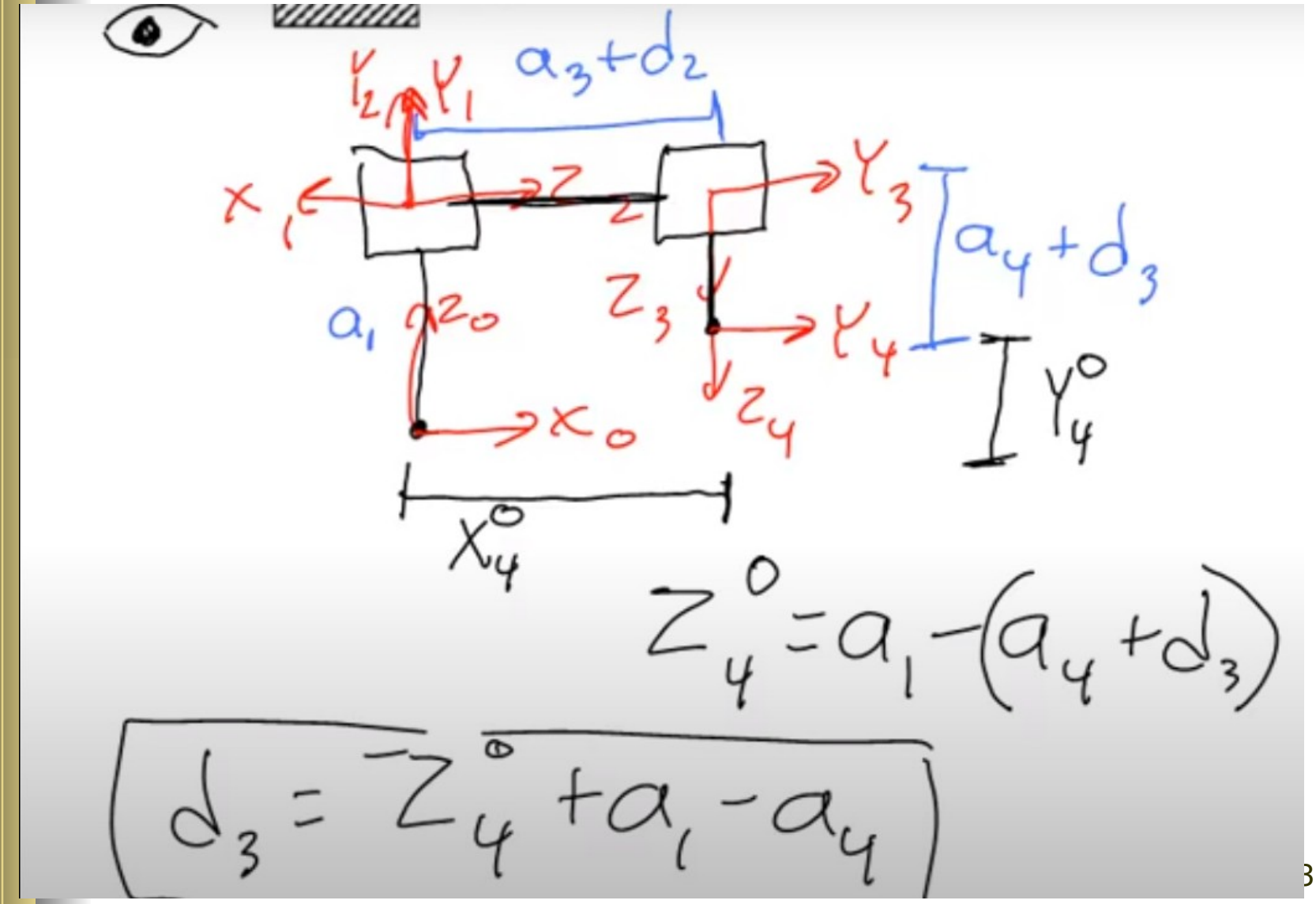

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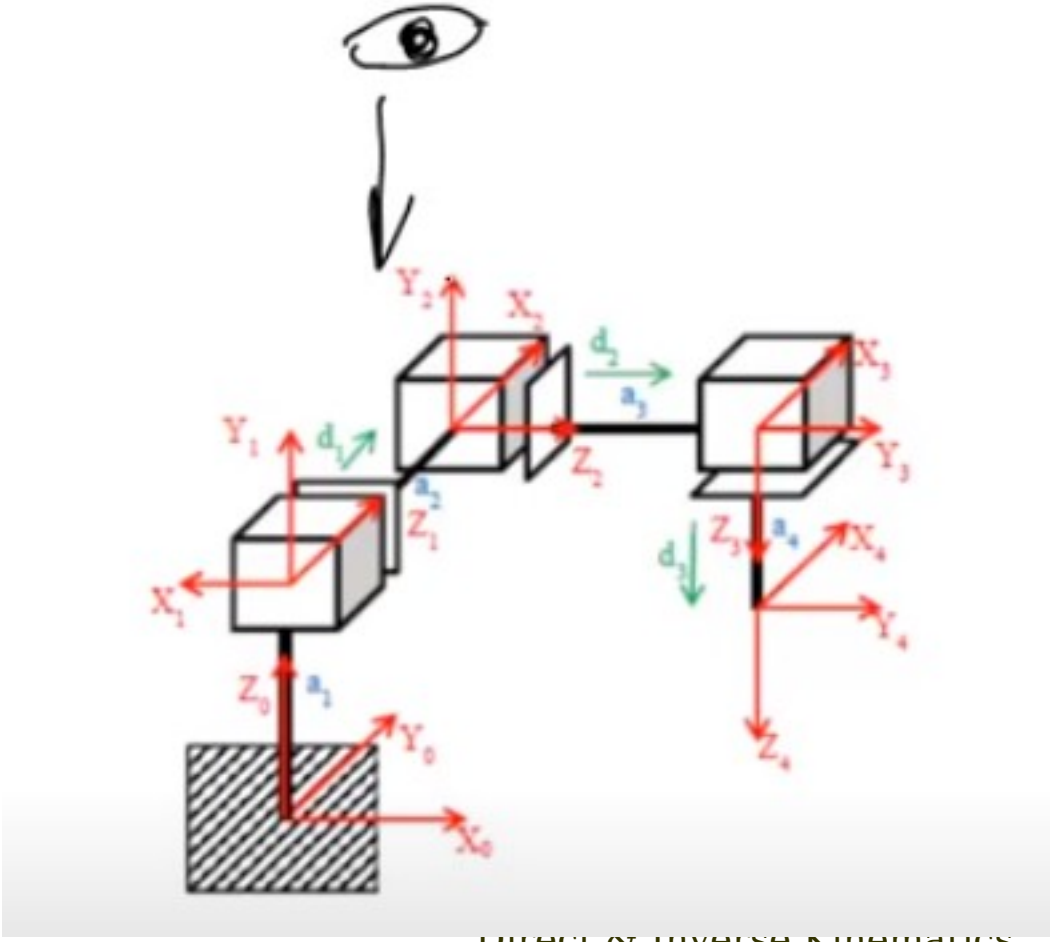
Now. draw the top view



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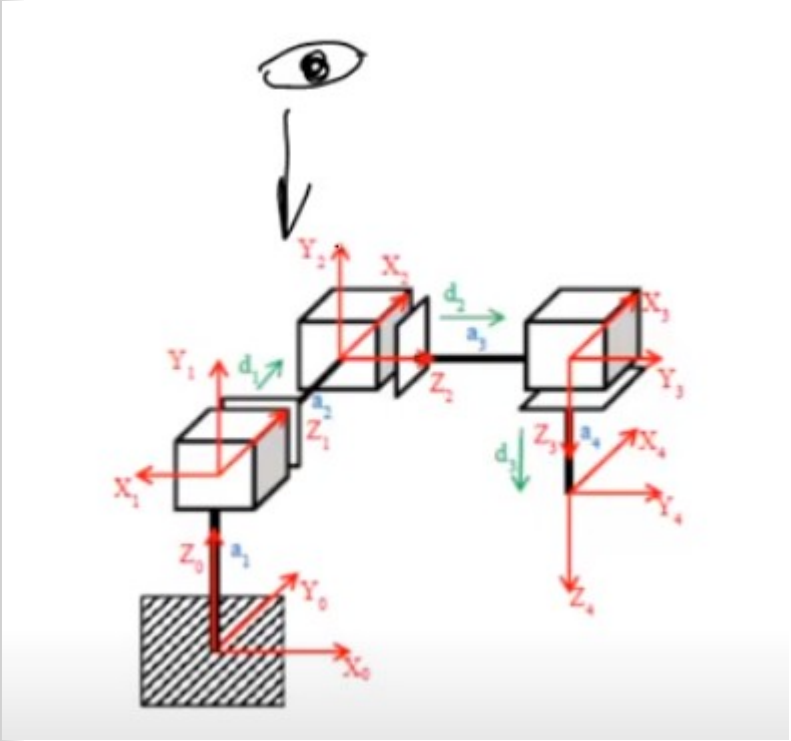
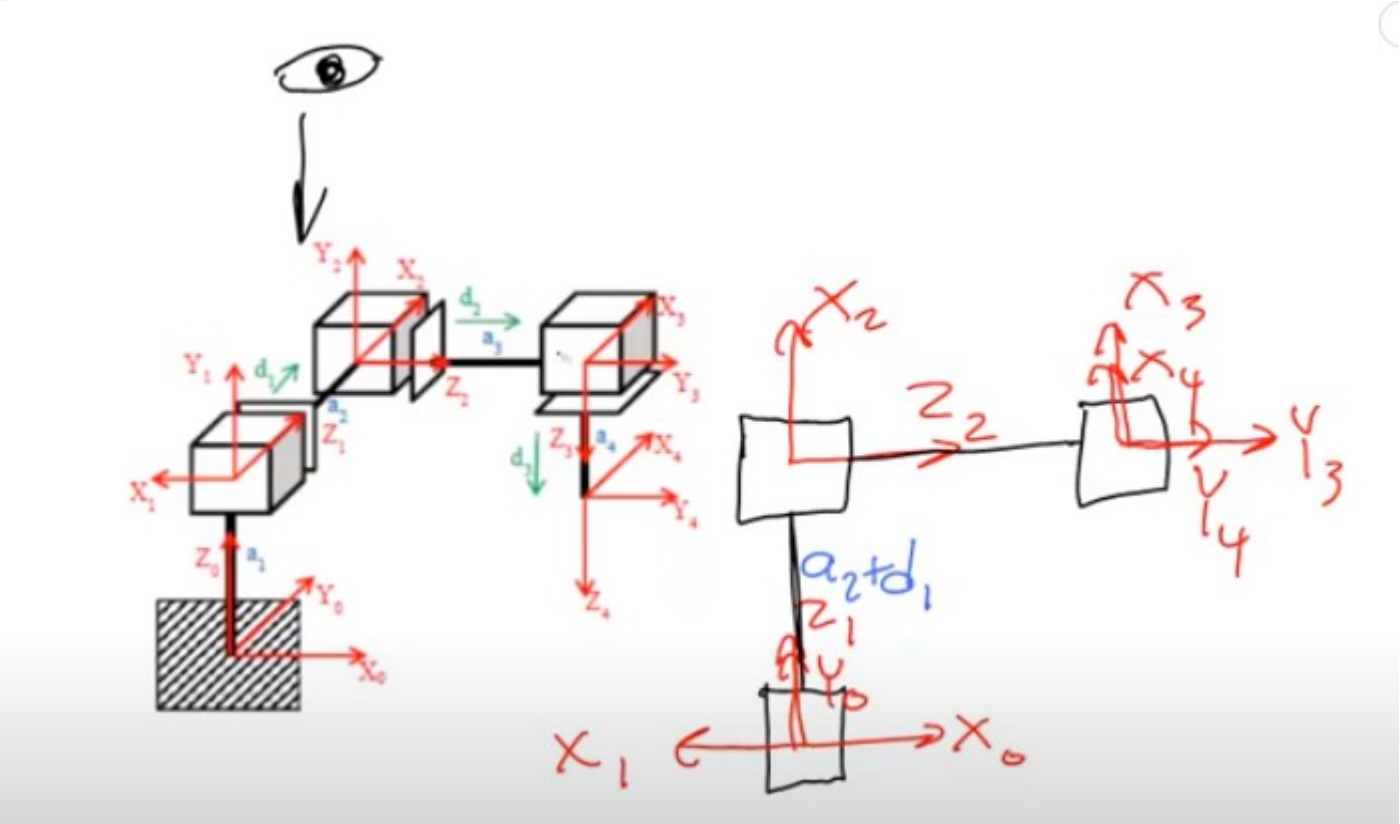


Example2: simple 3DOF manipulator

Find the X, Y and Z coordinates of the end effector in terms of the joint variable, then find the joint variables (d1, d2 and d3)in terms of these .coordinates

Can you find d1?



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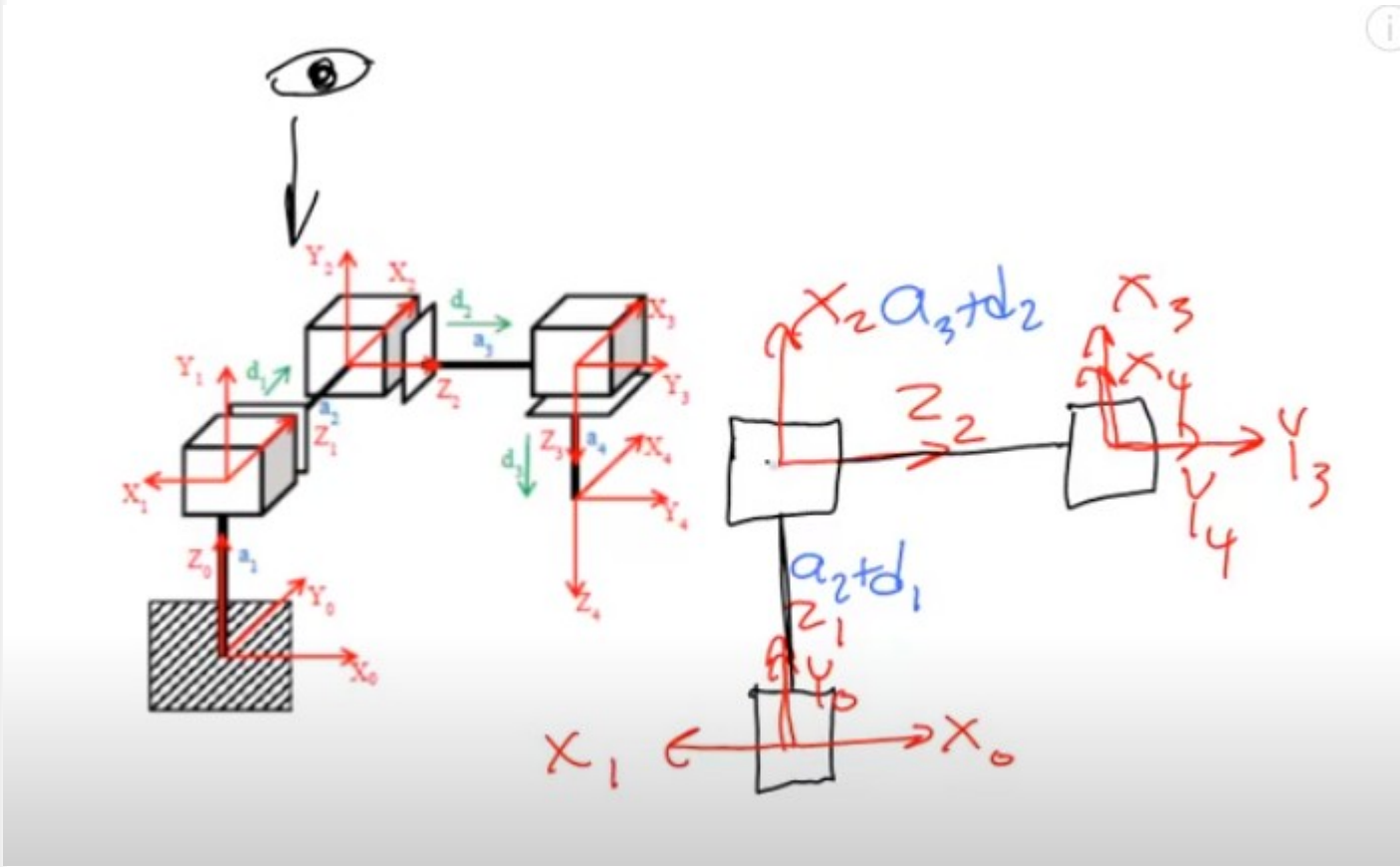

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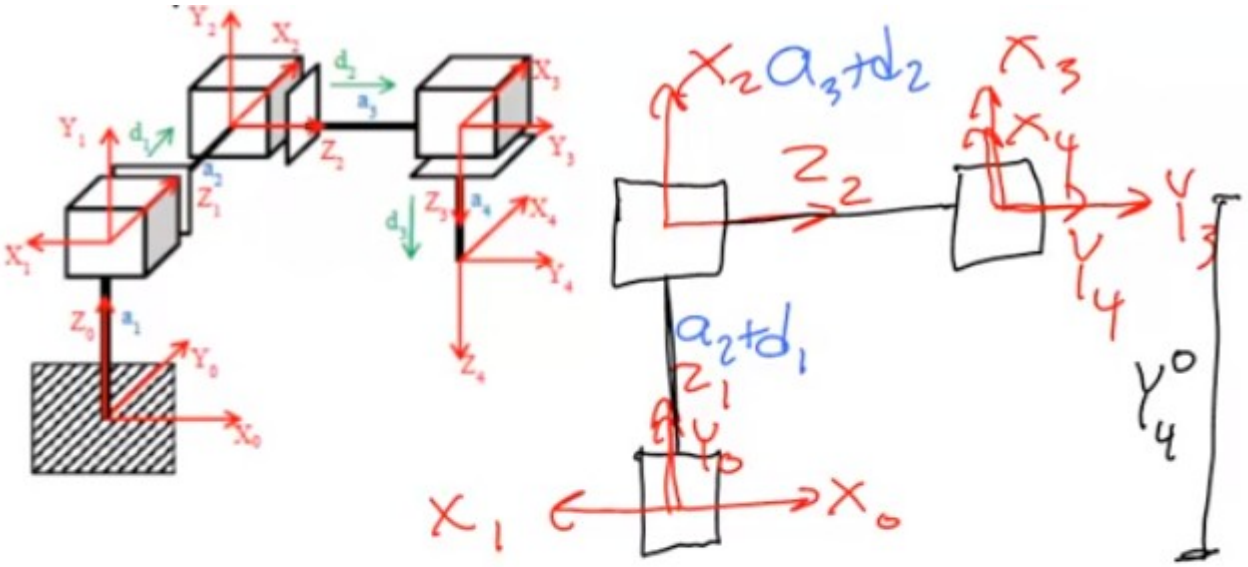
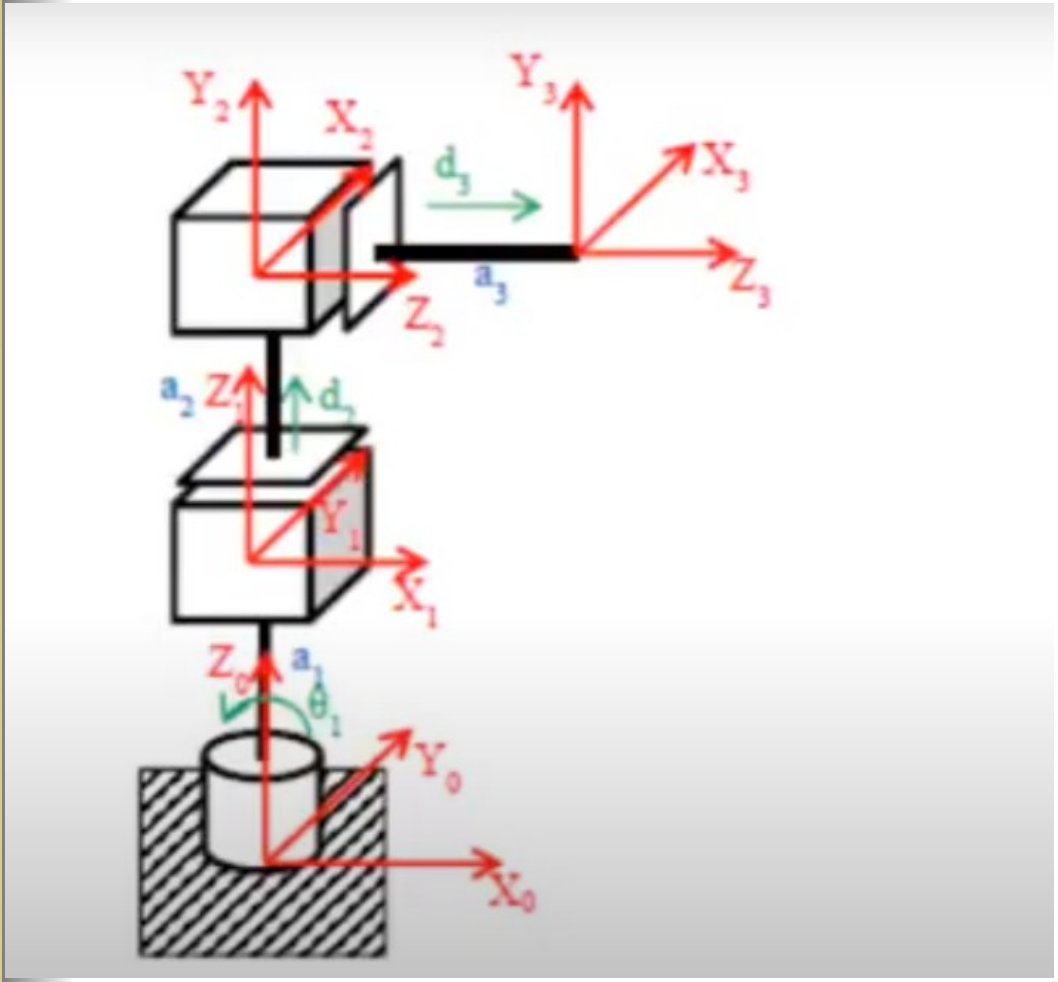


$$Y_4^0 = a_2 + d_1$$

$$d_1 = Y_4^0 - a_2$$

Example3: simple 3DOF manipulator

Find the X, Y and Z coordinates of the end effector in terms of the joint .variable, then find the joint variables in terms of these coordinates



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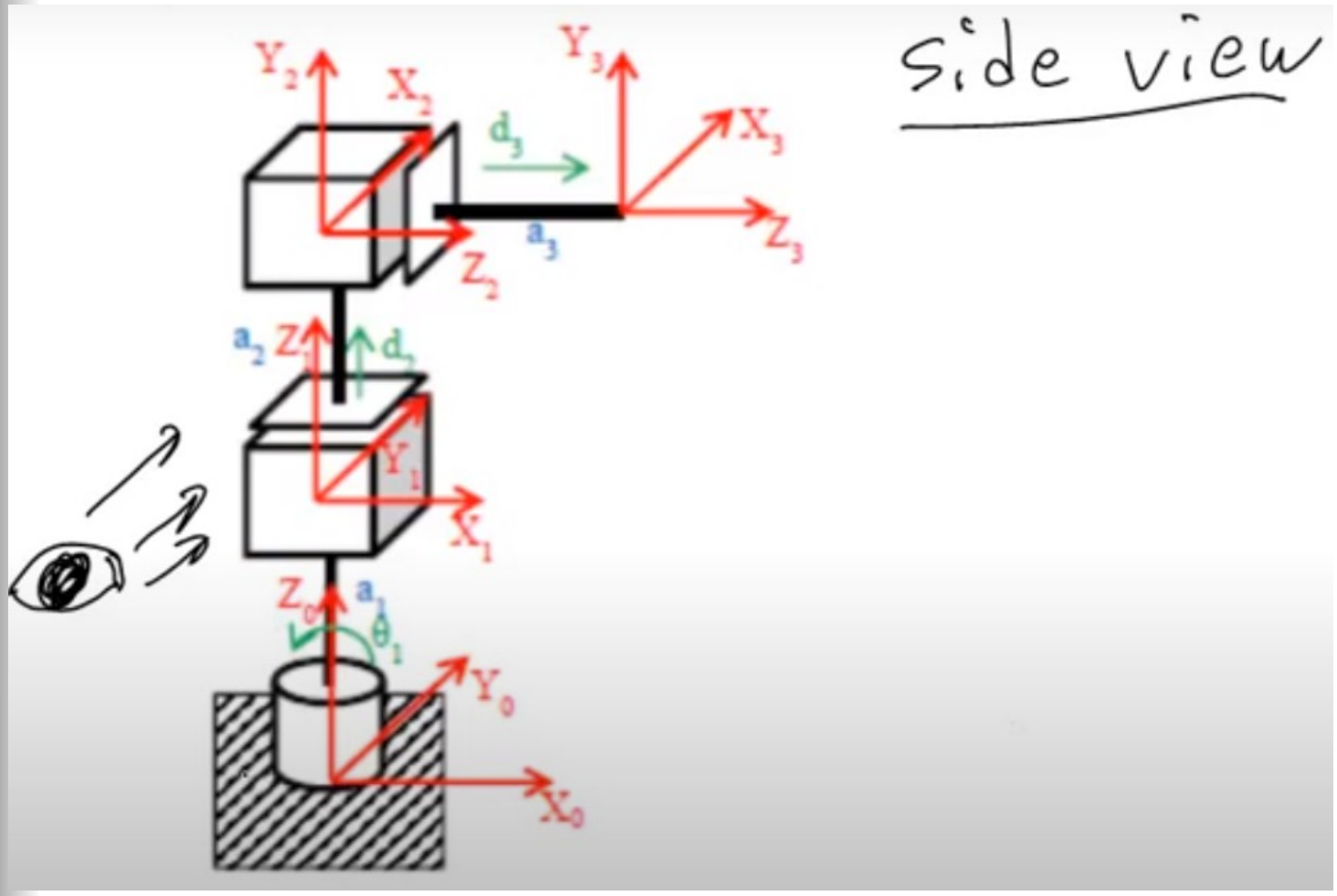
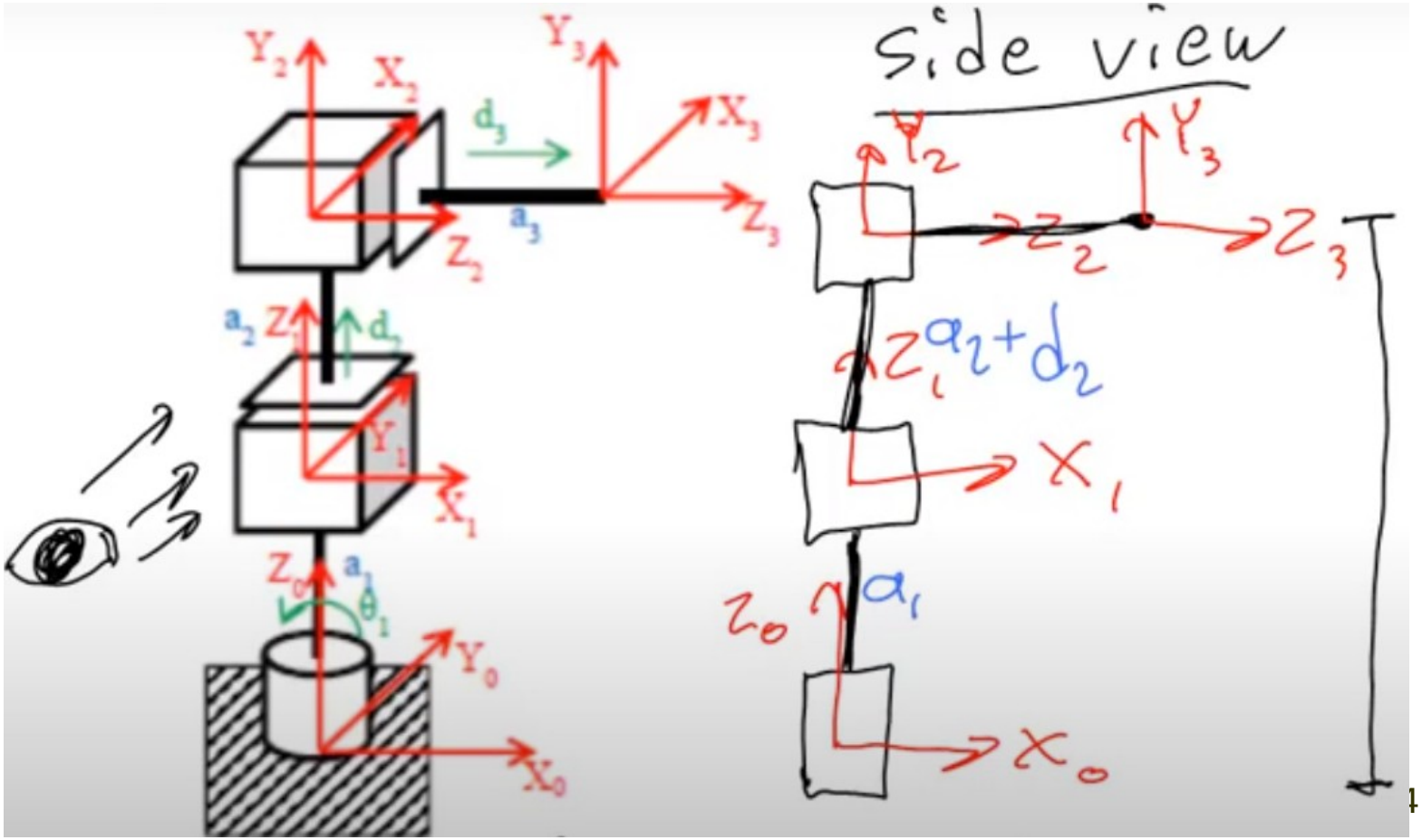
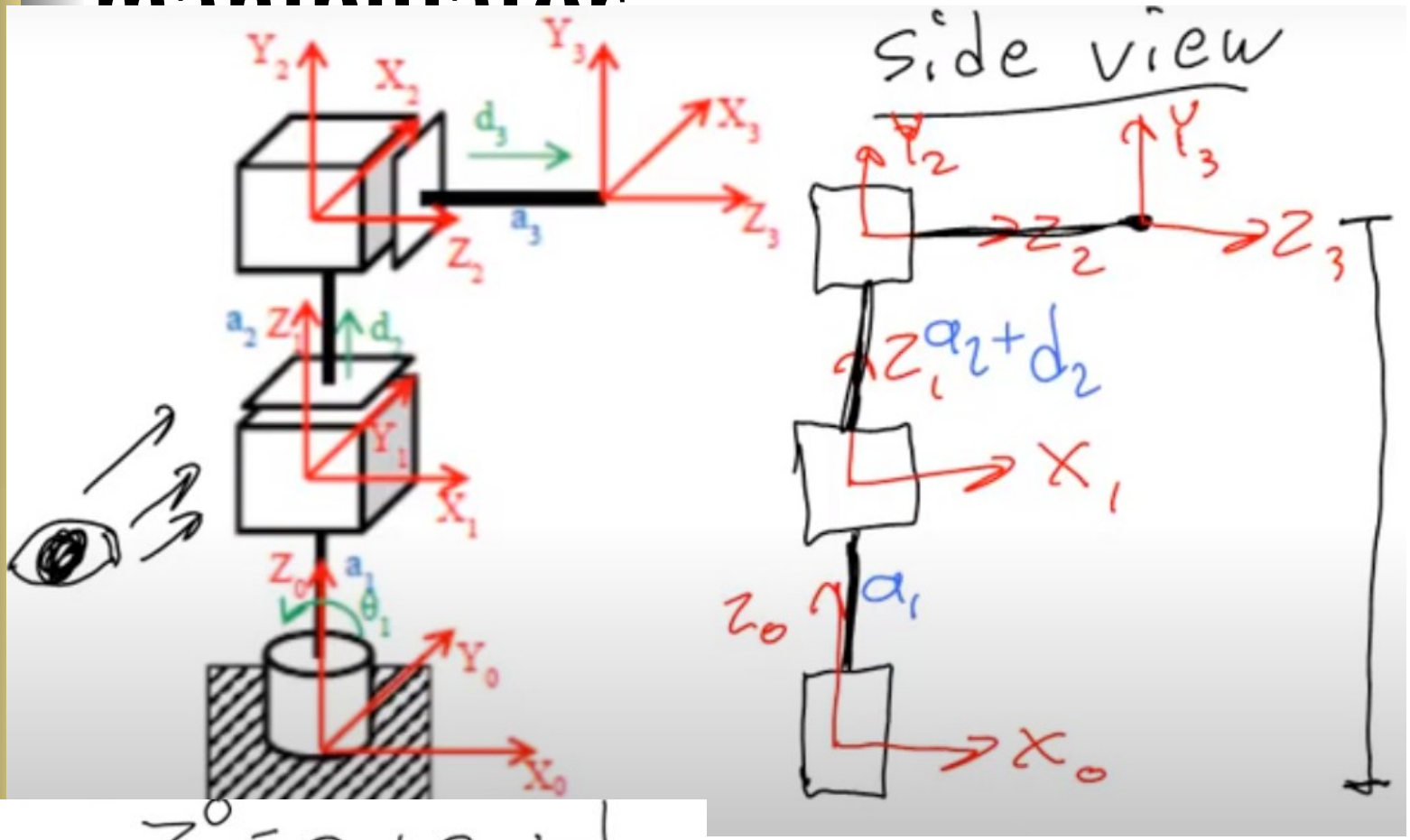


Example3: simple 3DOF manipulator

?Can you get an IK equation from this view



Example3: simple 3DOF manipulator



$$Z_3^0 = a_1 + a_2 + d_2$$

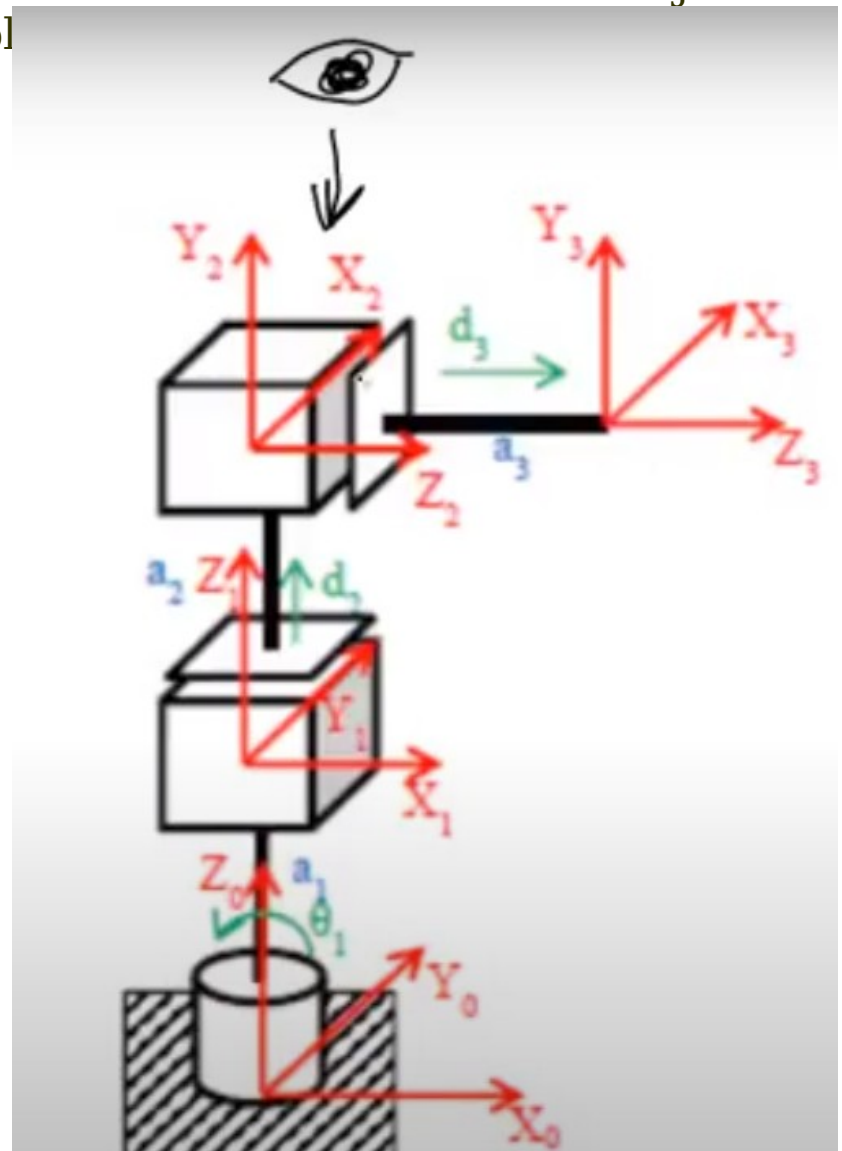
$$d_2 = Z_3^0 - a_1 - a_2$$

My first equation

Example3: simple 3DOF manipulator

Find the X, Y and Z coordinates of the end effector in terms of the joint .variable, then find the joint variabl

In the top view, make sure you consider a none zero rotation



Example3: simple 3DOF manipulator

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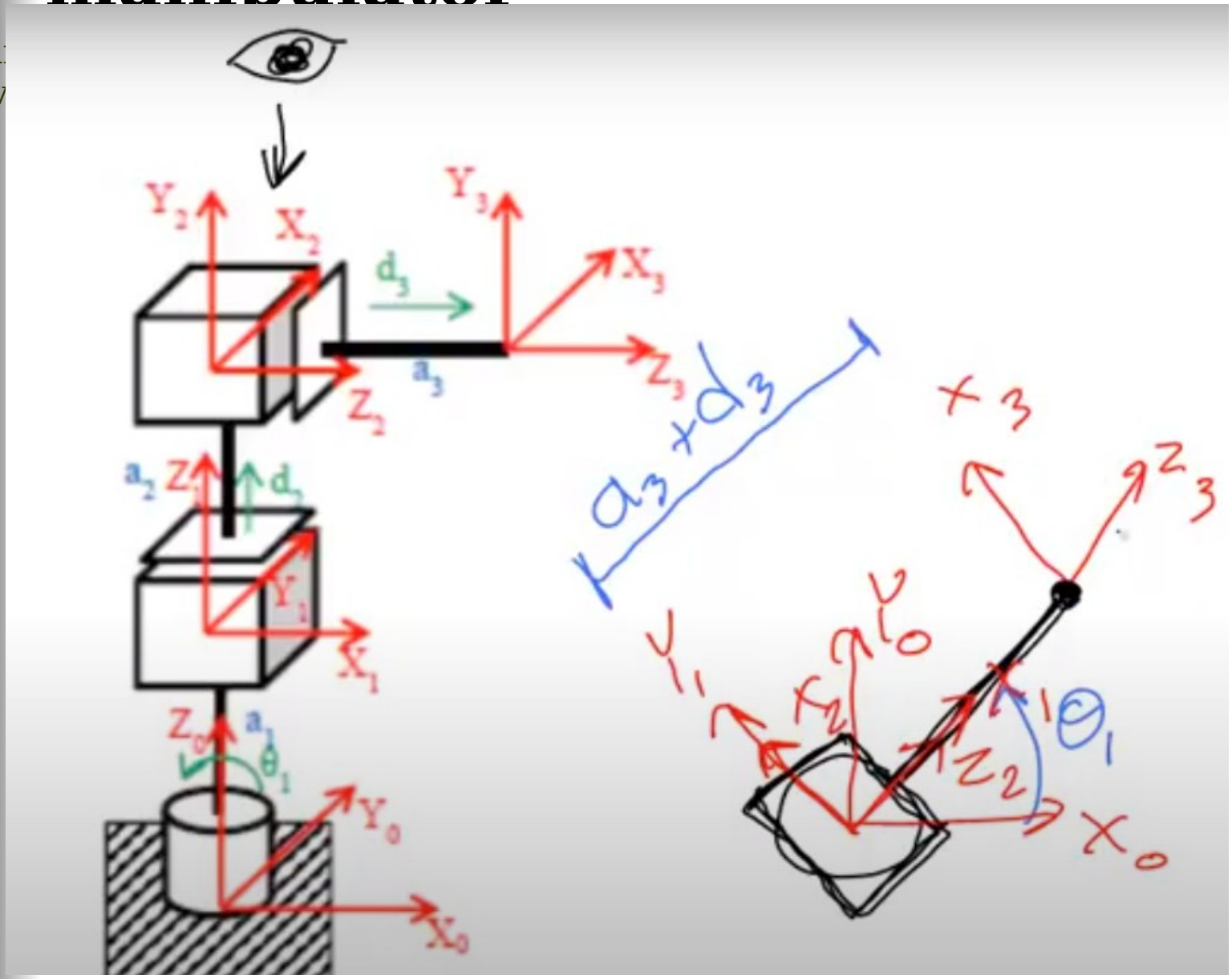
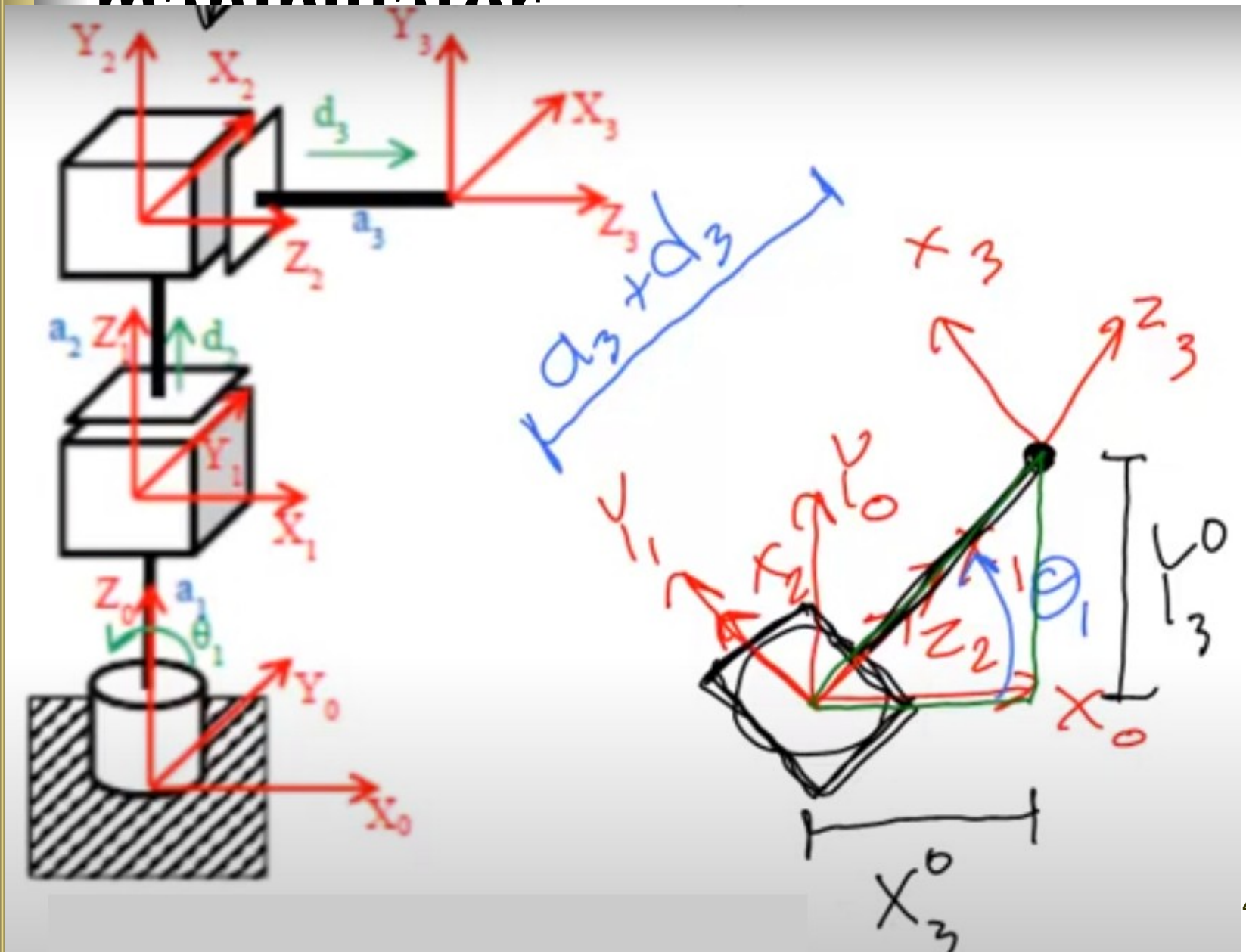


Example3: simple 3DOF manipulator



Example3: simple 3DOF manipulator

:Top view equatic

$$(X_3^0)^2 + (Y_3^0)^2 = (a_3 + d_3)^2$$

$$\sqrt{(X_3^0)^2 + (Y_3^0)^2} = a_3 + d_3$$

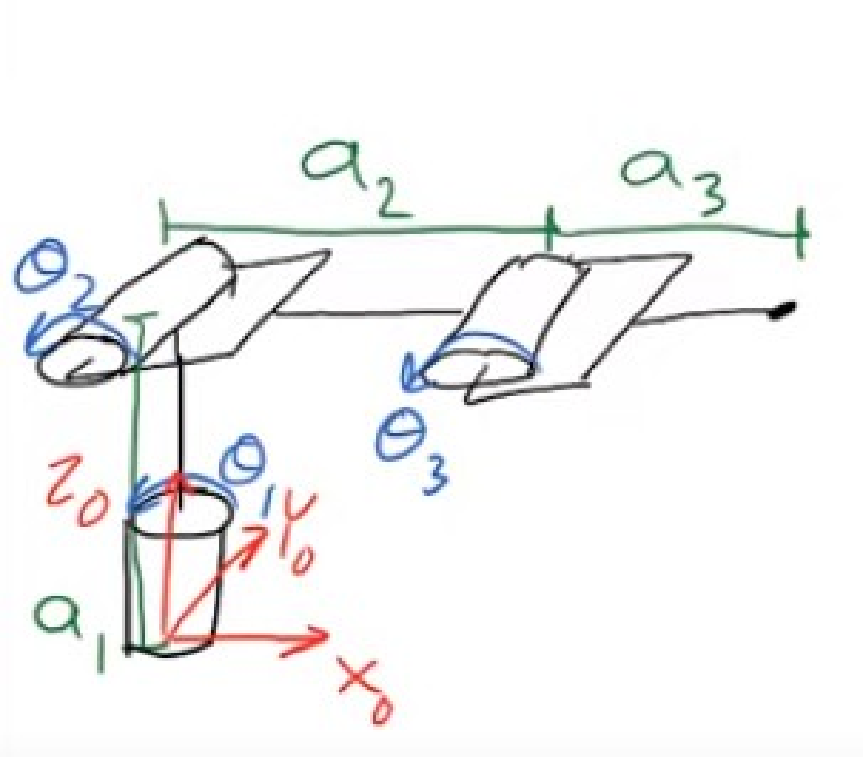
$$d_3 = \sqrt{(X_3^0)^2 + (Y_3^0)^2} - a_3$$

$$\tan\theta_1 = \frac{Y_3^0}{X_3^0}$$

$$\theta_1 = \operatorname{atan}\left(\frac{Y_3^0}{X_3^0}\right)$$

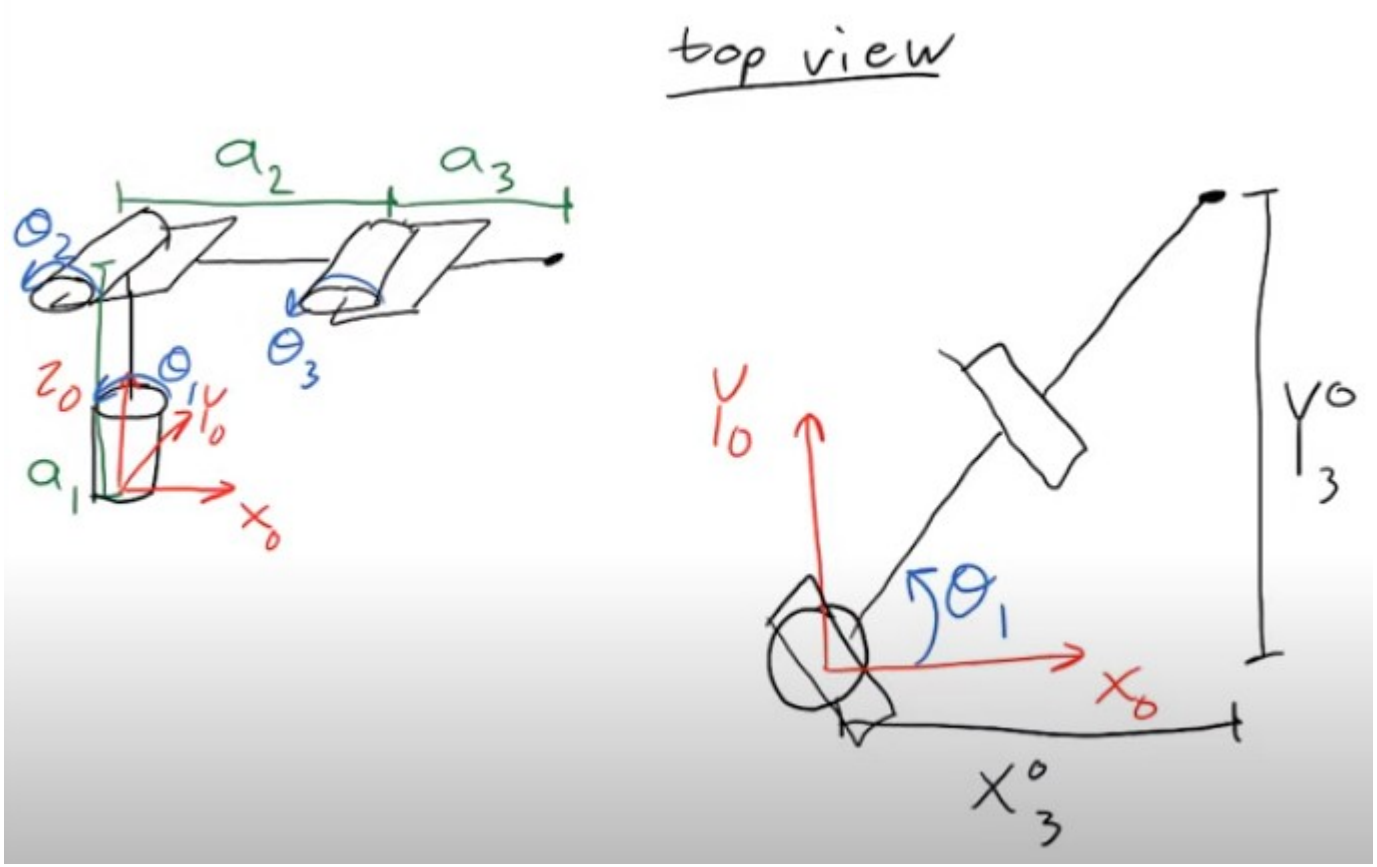
Example4: simple 3DOF manipulator

Find the X, Y and Z coordinates of the end effector in terms of the joint .variable, then find the joint variables in terms of these coordinates



Example3: simple 3DOF manipulator

Find the X, Y and Z coordinates of the end effector in terms of the joint .variable, then find the joint variables in terms of these coordinates



Example3: simple 3DOF manipulator

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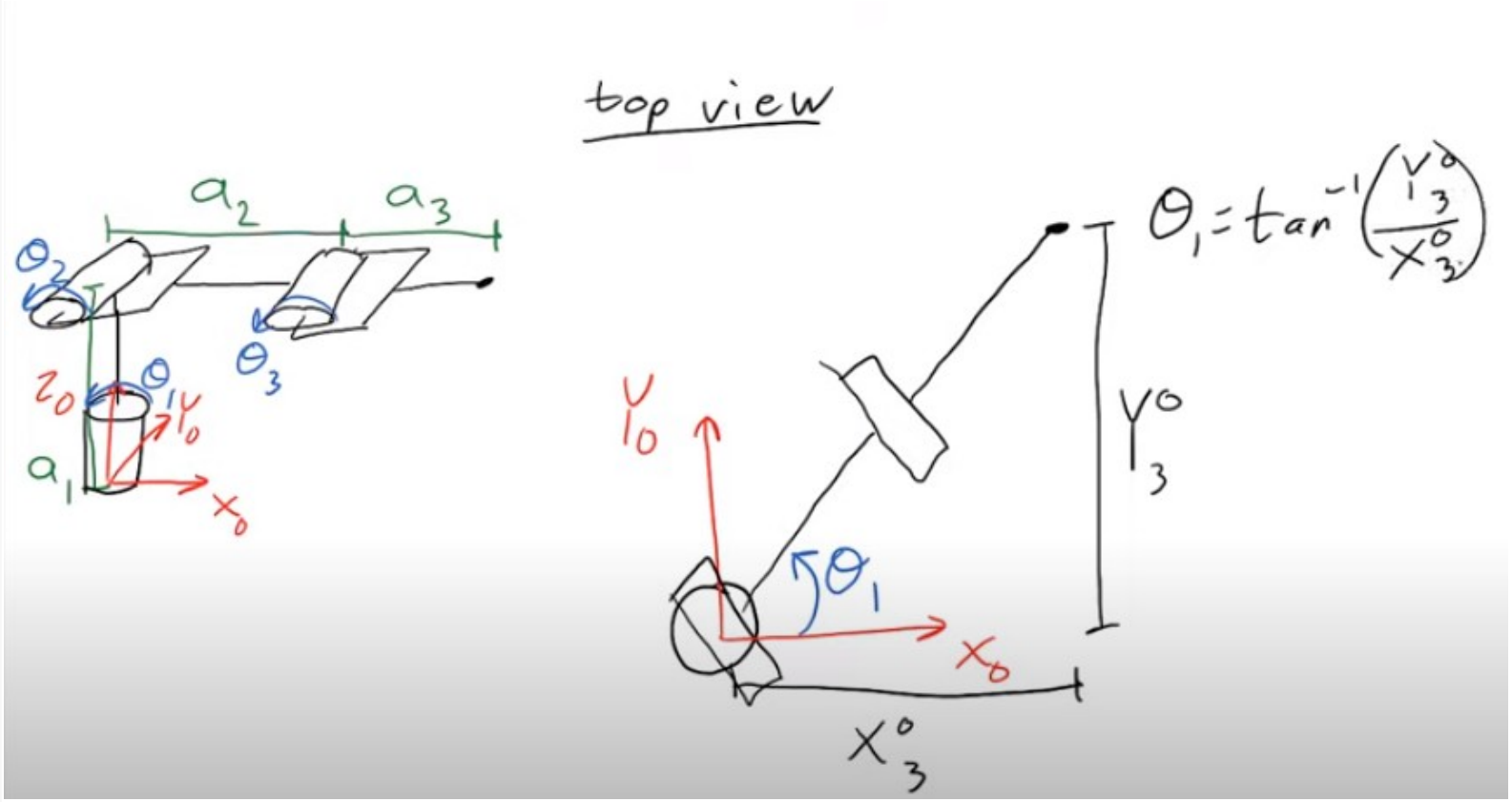
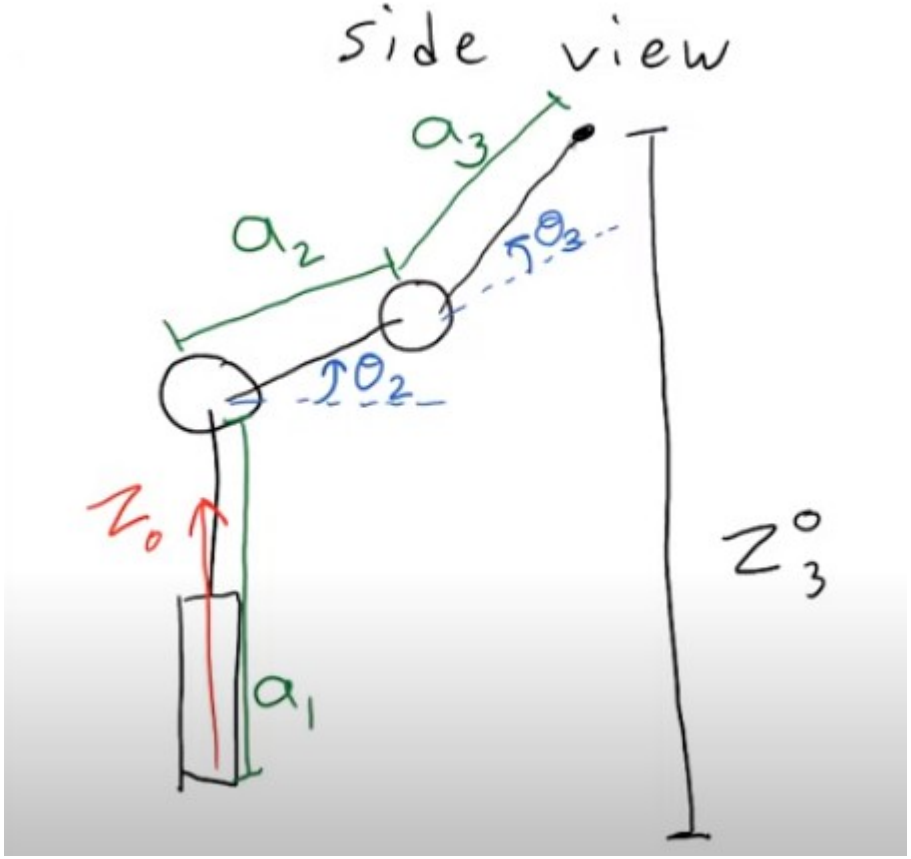


How about the side view?

Example3: simple 3DOF manipulator

Find the X, Y and Z coordinates of the end effector in terms of the joint .variable, then find the joint variables in terms of these coordinates

Can you know derive the other equations?



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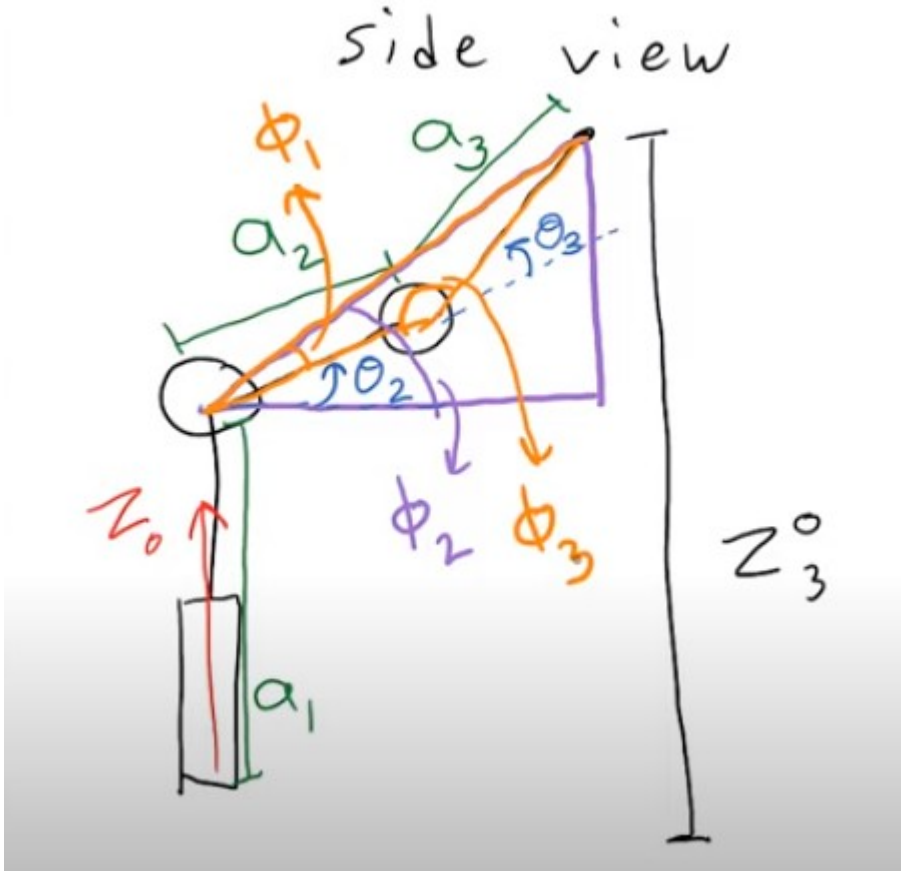

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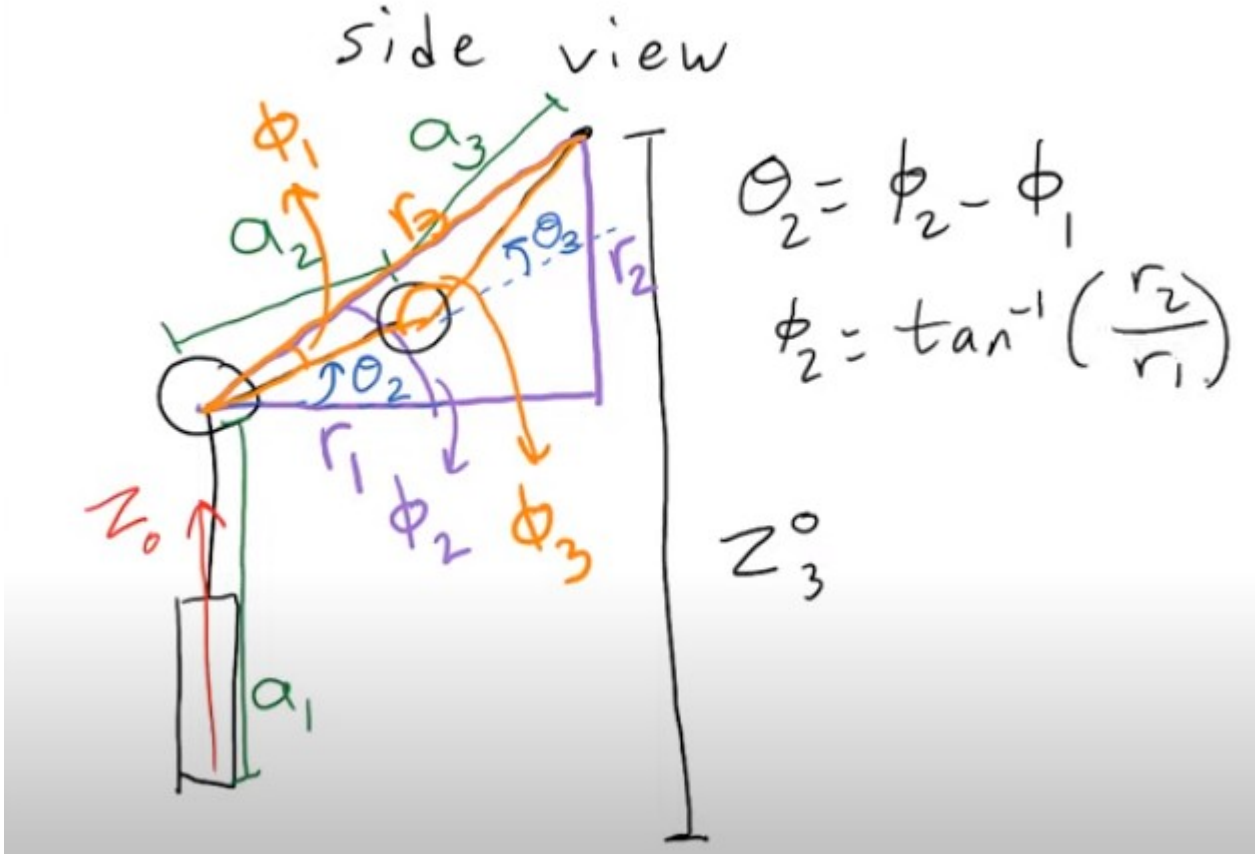

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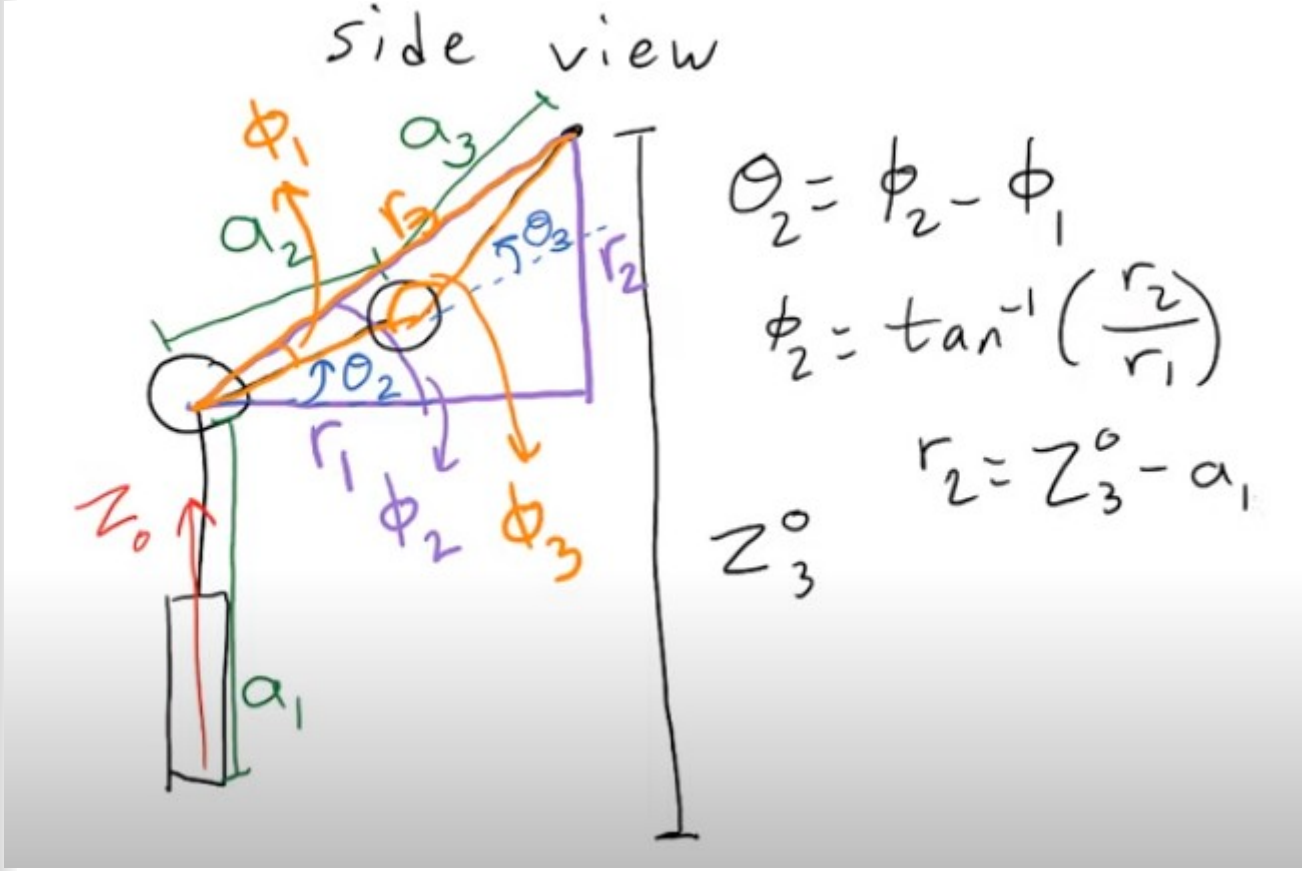
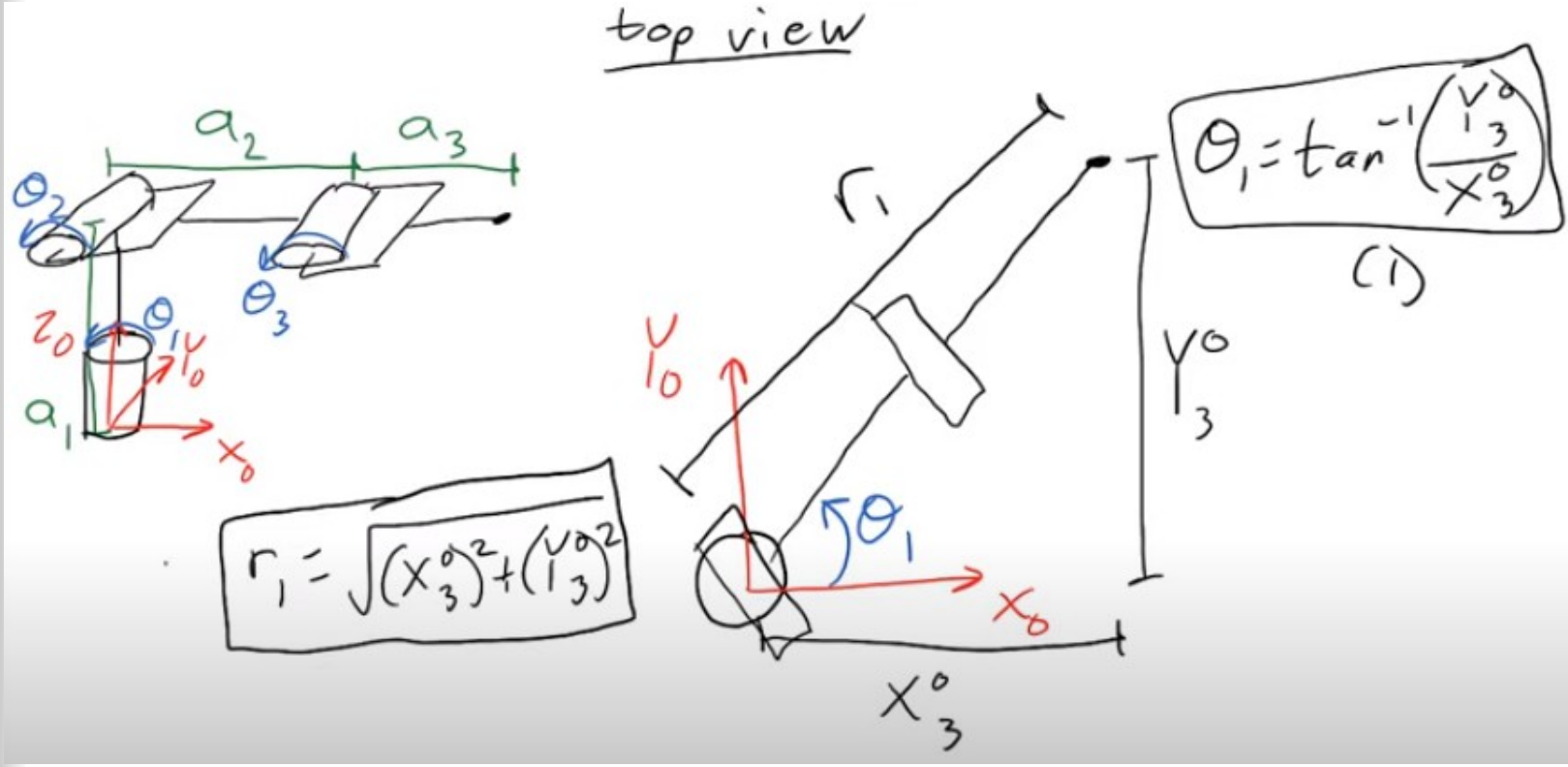


Example3: simple 3DOF manipulator

Find the X, Y and Z coordinates of the end effector in terms of the joint .variable, then find the joint variables in terms of these coordinates

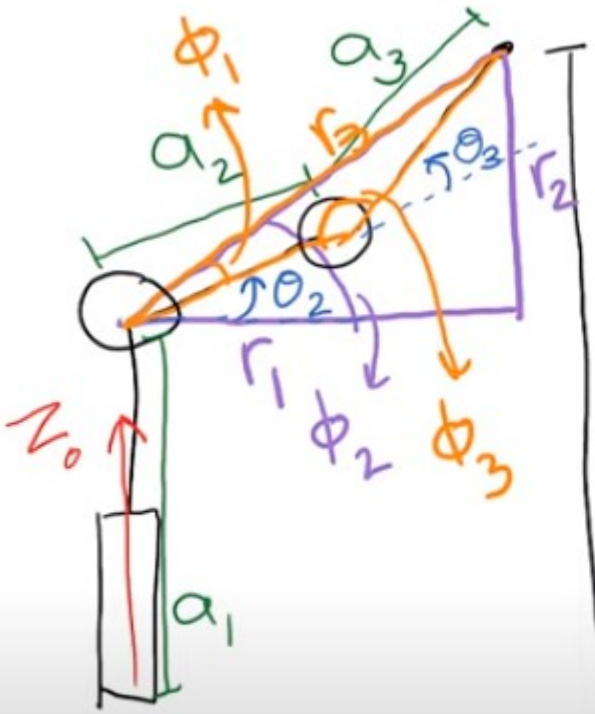
Back to the top view



Example3: simple 3DOF manipulator

Find the X, Y and Z coordinates of the end effector in terms of the joint .variable, then find the joint variables in terms of these coordinates

Back to the side view



$$\theta_2 = \phi_2 - \phi_1$$

$$\phi_2 = \tan^{-1}\left(\frac{r_2}{r_1}\right) \quad (4)$$

$$r_2 = Z_3^0 - a_1 \quad (3)$$

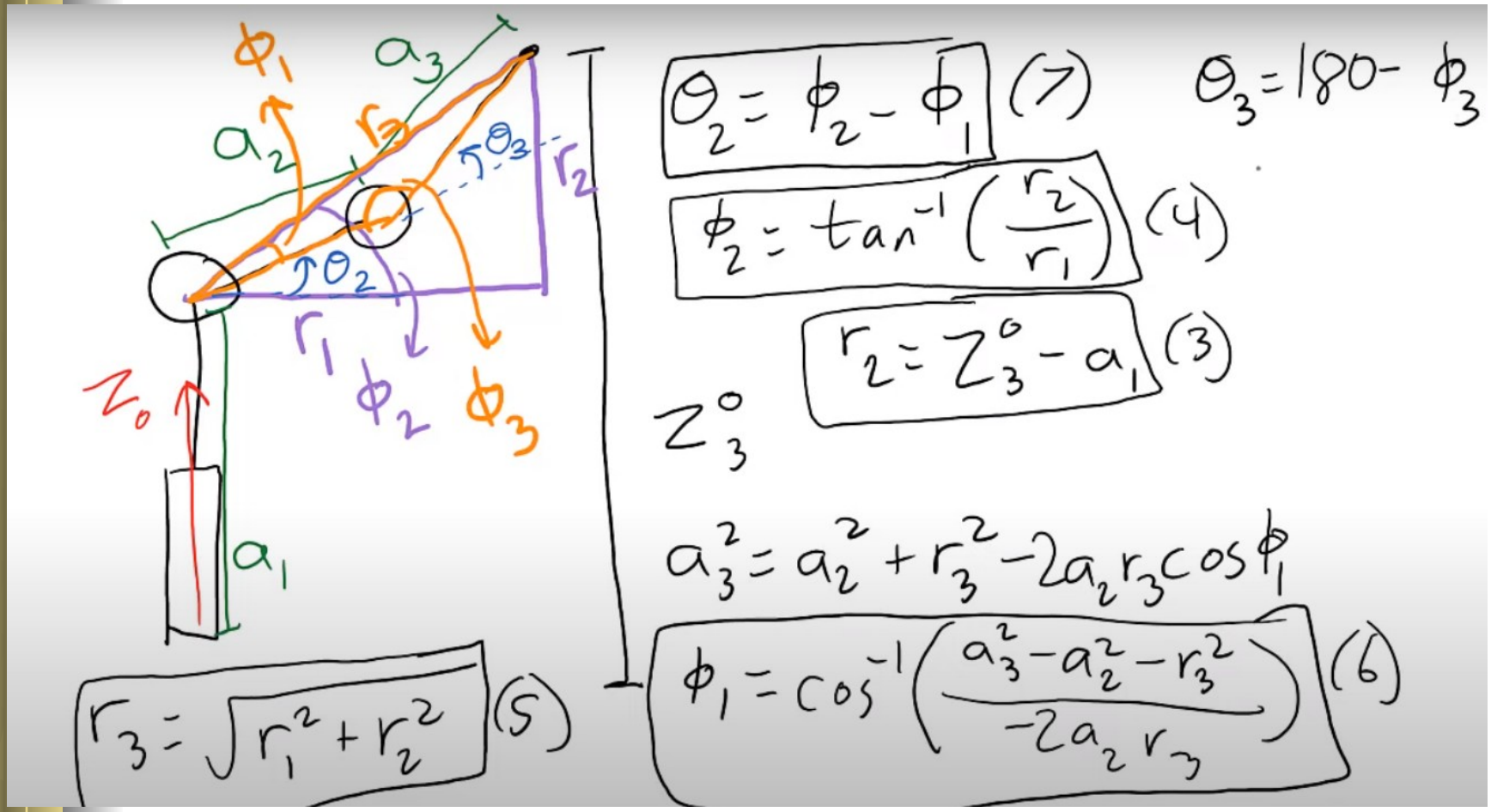
Z_3^0

$$a_3^2 = a_2^2 + r_3^2 - 2a_2 r_3 \cos\phi_1$$

$$\phi_1 = \cos^{-1}\left(\frac{a_3^2 - a_2^2 - r_3^2}{-2a_2 r_3}\right) \quad (5)$$

Example3: simple 3DOF manipulator

Find the X, Y and Z coordinates of the end effector in terms of the joint .variable, then find the joint variables in terms of these coordinates



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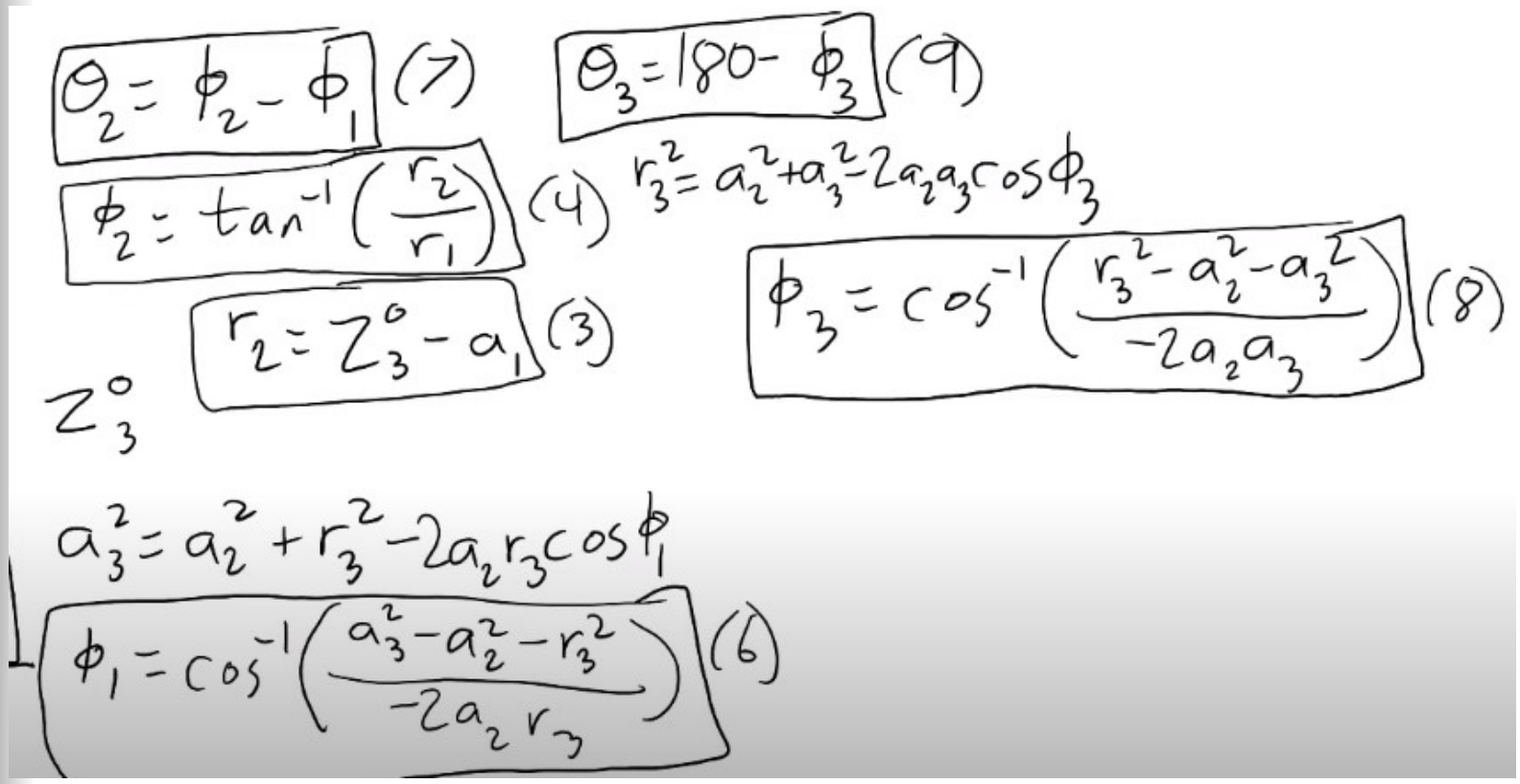
Find the X, Y and Z coordinates of the end effector in terms of the joint .variable, then find the joint variables in terms of these coordinates


$\theta_2 = \phi_2 - \phi_1$ (7)
$\theta_3 = 180 - \phi_3$ (9)
$\phi_2 = \tan^{-1}\left(\frac{r_2}{r_1}\right)$ (4)
$r_3^2 = a_2^2 + a_3^2 - 2a_2a_3\cos\phi_3$
$\phi_3 = \cos^{-1}\left(\frac{r_3^2 - a_2^2 - a_3^2}{-2a_2a_3}\right)$ (8)
$r_2 = Z_3^0 - a_1$ (3)
Z_3^0
$a_3^2 = a_2^2 + r_3^2 - 2a_2r_3\cos\phi_1$
$\phi_1 = \cos^{-1}\left(\frac{a_3^2 - a_2^2 - r_3^2}{-2a_2r_3}\right)$ (6)

Example3: simple 3DOF manipulator

Find the X, Y and Z coordinates of the end effector in terms of the joint .variable, then find the joint variables in terms of these coordinates

Thank you